

DYNAMICS OF SYMMETRY BREAKING IN FRW COSMOLOGIES: EMERGENCE OF SCALING

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Abstract

The dynamics of a symmetry breaking phase transition is studied in a radiation and matter dominated spatially flat FRW cosmology in the large N limit of a scalar field theory. The quantum density matrix is evolved from an initial state of quasiparticles in thermal equilibrium at a temperature higher than the critical. The cosmological expansion decreases the temperature and triggers the phase transition. We identify three different time scales: an early regime dominated by linear instabilities and the exponential growth of long-wavelength fluctuations, an intermediate scale when the field fluctuations probe the broken symmetry states and an asymptotic scale wherein a **scaling regime** emerges for modes of wavelength comparable to or larger than the horizon. The scaling regime is characterized by a dynamical physical correlation length $\xi_{phys} = d_H(t)$ with $d_H(t)$ the size of the causal horizon, thus there is one correlated region per causal horizon. Inside these correlated regions the field fluctuations sample the broken symmetry states. The amplitude of the long-wavelength fluctuations becomes non-perturbatively large due to the early times instabilities and a semiclassical but *stochastic* description emerges in the asymptotic regime. In the scaling regime, the power spectrum is peaked at zero momentum revealing the onset of a Bose-Einstein condensate. The scaling solution results in that the equation of state of the scalar fields is the *same as that of the background fluid*. This implies a Harrison-Zeldovich spectrum of scalar density perturbations for long-wavelengths. We discuss the corrections to scaling as well as the universality of the scaling solution and the differences and similarities with the classical non-linear sigma model.

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I. INTRODUCTION, MOTIVATION AND SUMMARY

Symmetry breaking phase transitions play a fundamental role in the Standard Model of particle physics and its generalizations and are conjectured to have occurred in the early Universe. Some versions of inflationary models invoke a symmetry breaking phase transition to generate the inflationary stage [1] and the process of field ordering after the phase transition has been argued to produce density fluctuations that could seed galaxy formation [1,2]. Certainly, symmetry breaking phase transitions in the early Universe are very important at the electroweak scale and at much lower temperatures for the hadronization and chiral phase transitions in QCD. At a more speculative level, late time phase transitions in the matter dominated era had been proposed [3] with the possibility that pseudo Nambu-Goldstone bosons may contribute to the dark matter component and influence large scale structure formation.

Typically, symmetry breaking phase transitions are studied by means of an effective potential which is the free energy density for a space-time constant expectation value of the order parameter. Although this formulation provides a good qualitative understanding of the *equilibrium* properties it is unable to address the important issue of the *dynamics* of symmetry breaking and the non-equilibrium process of phase ordering and phase separation. The simple picture of the order parameter rolling down the potential hill although intuitively appealing does not capture the correct dynamics. A consistent non-equilibrium field theoretical treatment reveals that the quantum fluctuations are extremely important and enhanced by the instabilities that are the hallmark of the phase transition [4].

The picture that emerges from consistent studies of the dynamics of phase transitions in Minkowski space time [4–7] is remarkably similar to phase ordering kinetics in a symmetry breaking phase transition in condensed matter systems [8–10].

Recent results in quantum field theory [6,7] in Minkowski space-time reveal the following picture: when the system is cooled below the critical temperature on time scales shorter than the relaxation time of long-wavelength fluctuations, the process of phase separation begins via linear spinodal instabilities and the exponential growth of these long-wavelength fluctuations [4,6,7]. A new dynamical time scale emerges at which the back-reaction of these fluctuations becomes of the order of the tree level term and the linear instabilities shut-off. At this time scale, the fluctuations of the field sample the broken symmetry equilibrium states (minima of the free energy) which in turn implies that the amplitude of the long-wavelength fluctuations becomes non-perturbatively large [4,6,7] and consistent non-perturbative methods are necessary to study the real-time dynamics of the phase transition. When the backreaction from the field fluctuations becomes comparable to the tree level terms in the equations of motion the non-linearities become very important. In this late time regime a dynamical length scale emerges $\xi(t) = t$ which is a consequence of causality and determines the size of the correlated regions, the emergence of a scaling behavior and the onset of a novel non-equilibrium condensate [6–8,11]. Recently a very thorough study of the non-equilibrium time evolution has been provided in finite volume (finite momentum resolution) with non-perturbative techniques providing a consistent picture of the dynamics of phase separation [11].

In cosmology the non-equilibrium dynamics of a symmetry breaking phase transition has been studied within the context of **inflationary** scenarios including quantum backreac-

tion from the matter fields in the metric dynamics, and non-equilibrium aspects of particle production and reheating including quantum backreaction in the large N limit had been studied in Friedmann-Robertson-Walker (FRW) cosmologies in a fixed background [13,14]. We investigate here the *dynamics of a phase transition and phase ordering kinetics* in a radiation and in a matter dominated FRW background metric from a quantum field theory formulation that includes self-consistently the quantum and thermal fluctuations in the dynamics.

Phase ordering dynamics in a FRW background has been previously studied in the large N limit of the *classical non-linear* sigma model in [15,16]. Detailed numerical and analytical studies of the classical non-linear sigma model revealed the emergence of a scaling solution of the classical field equations that reflect the dynamics of Goldstone bosons in the broken symmetry state [15]. The scaling solution allowed to study the cosmological perturbations seeded by the fluctuations of the Goldstone scaling field [15,16] and provides a solvable example to study the evolution of causal cosmological perturbations. The importance of a scaling solution lies in the fact that for causal perturbations the dynamical range of time scales between the phase transition and matter-radiation decoupling is very large to be tractable numerically, the scaling property of correlation function thus allows to extrapolate them to arbitrarily long time scales [2,16].

The *non-linear sigma model* is a theory that describes the dynamics of Goldstone bosons in a broken symmetry state and as such the field is constrained to the vacuum manifold. Hence this model *does not* allow to study the question of the dynamics of symmetry breaking phase transitions and the non-equilibrium aspects of phase separation and ordering beginning from an initial high temperature state in the unbroken symmetry phase and cooling through the phase transition until correlated regions of broken symmetry are formed and grow.

The goals: The goals and focus of this article are to study the important issue of the *dynamics of the phase transition* which simply cannot be addressed within the context of the non-linear sigma model. Namely, i) the non-equilibrium dynamics of a symmetry breaking phase transition from an initial high temperature unbroken phase to a final low temperature broken symmetry phase in a radiation or matter dominated background FRW cosmology, ii) to follow the process of phase separation and ordering directly in real time from the early time instabilities to the formation of correlated regions, iii) to study the emergence of a scaling regime as a dynamical consequence of the phase transition and iv) implications for the equation of state of the matter described by the quantum scalar field. Furthermore we analyze the deviations from scaling and discuss the necessity for scaling violations for self-consistency. The description of cosmological perturbations is outside the scope of this paper. We keep our focus on the description of the non-equilibrium dynamics of the phase transition and the resulting consequences.

The model: As argued above a consistent study of the dynamics of a phase transition and the process of phase separation requires a non-perturbative framework. There are very few non-perturbative frameworks that are i) renormalizable ii) maintain all of the conservation laws, iii) lend themselves to a detailed analytical or numerical study and iv) can be consistently improved. We thus study the *quantum linear* sigma model of a scalar field in the vector representation of $O(N)$ in the leading order in the large N limit in a fixed spatially flat radiation and or matter dominated FRW cosmological background. There are several noteworthy features of this model that deserve comparison with the classical non-

linear sigma model studied in detail in [15,16]: i) we consider the full quantum field theory as described by a time dependent quantum density matrix prepared initially in equilibrium at high temperature, ii) unlike the non-linear sigma model, this is a renormalizable quantum field theory and the ultraviolet divergences are absorbed in renormalizations of masses, couplings and field amplitudes, iii) the linear sigma model allows to go from the unbroken to the broken symmetry phase, the non-linear model is only defined in the broken symmetry phase.

The strategy: The main ingredient in the description of the non-equilibrium dynamics is the time evolution of an initially prepared quantum density matrix. We begin our analysis by implementing the large N limit, which to leading order leads to a self-consistent quadratic Hamiltonian with effective frequencies that depend on time through the background scale factor as well as through the self-consistent mean field. The density matrix is taken to be Gaussian consistent with the leading order in the large N and its initial form describes a state in which the self-consistent quasiparticles are in local thermal equilibrium at an initial temperature $T_i > T_c$ with T_c the critical temperature. We will not justify the choice of an initial thermal state for the self-consistent quasiparticles, and we will simply *assume* that such initial state is physically reasonable and is an acceptable description of the system prior to the phase transition. This initial density matrix evolves in time through the Liouville equation which has an exact solution in the leading order in the large N limit. The advantage of our formulation in terms of the quantum density matrix is that it leads to a clear description of the emergence of (semi) classical but stochastic description. The expansion of the cosmological background results in adiabatic cooling that triggers the phase transition resulting in long-wavelength instabilities which are the hallmark of the process of phase separation and ordering and completely dominate the early time evolution of correlations. The early time dynamics can be understood analytically since for weak coupling the backreaction is perturbatively small and the equations are essentially linear. The intermediate and asymptotic dynamics are studied analytically and numerically both in radiation and matter dominated FRW backgrounds.

Summary of results: The main results of this article can be summarized as follows:

- The dynamics of the phase transition reveals several different time scales: an early time scale dominated by the exponential growth of long-wavelength fluctuations as a consequence of the instabilities associated with the phase transition. An intermediate time scale at which the self-consistent backreaction from the quantum and thermal fluctuations begins to compete with the tree-level terms in the equations of motion and signals the onset of non-perturbative and non-linear dynamics. An asymptotic time scale and the emergence of a scaling solution for fluctuations with wavelengths of the order of or larger than the causal horizon. Corrections to scaling are an unavoidable consequence of the dynamics and they are important for self-consistency, their form is uniquely determined by the background.
- In the scaling regime there emerges a dynamical length scale $\xi_{phys}(t) = d_H(t)$ with $d_H(t)$ the size of the causal horizon. This length scale determines the size of the correlated regions, inside which the field fluctuations probe the broken symmetry states. There is one such correlated region per causal horizon. This is a microscopic, quantum field theoretical justification of Kibble's original proposal [2,17]. The power spectrum

is peaked at superhorizon wavelengths and signals the onset of a non-equilibrium condensate.

- After the intermediate time regime the amplitude of long-wavelength fluctuations become non-perturbatively large and the phases freeze out. A semiclassical but *stochastic* description emerges and field configurations with large amplitudes and long-wavelengths are represented in the ensemble (density matrix) with non-negligible probability.
- As a consequence of the scaling solution, the equation of state for the fluid of the field fluctuations is the *same* as that of the background. As a consequence, the density fluctuations $\delta\rho/\rho_{background}$ are time independent. Therefore, superhorizon models will exhibit a Harrison-Zeldovich spectrum.
- The universality of the scaling solution and the corrections to scaling are discussed in detail.

The article is organized as follows: In section II we set up the model, implement the large N limit to leading order and solve the equations that determine the time evolution of the initially prepared density matrix. In section III we study analytically and numerically the dynamics of symmetry breaking, establishing the different time scales, the emergence of scaling and of a semiclassical description. In section IV the consequences of the scaling solution are analyzed: the dynamical length scale and the correlated regions and the equation of state of the scalar fields fluctuations. Section V provides a comparison between our results and those obtained for the *classical* $O(N)$ non-linear sigma model. Section VI summarizes our conclusions.

II. DENSITY MATRIX EVOLUTION IN THE LARGE N LIMIT

We start by setting up the formulation of the time evolution of the quantum density matrix in a spatially flat FRW cosmology with a *fixed* background metric. In comoving coordinates the metric element is given by

$$ds^2 = dt^2 - a^2(t) d\vec{x}^2 \quad (2.1)$$

with

$$H^2(t) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3M_{Pl}^2} \rho(t) \quad (2.2)$$

$$\rho(t) = \frac{\rho_R^i}{a^4(t)} + \frac{\rho_M^i}{a^3(t)} \quad (2.3)$$

and $\rho(t)$ is a *fixed* background energy density which is taken to be a combination of radiation and matter with ρ_R^i , ρ_M^i the energy densities in radiation and matter respectively at the initial time $t = t_0$ with $a(t_0) = 1$.

We consider a theory of N scalar fields in the vector representation of $O(N)$ in the leading order in the large N limit. The action is given by

$$S = \int d^4x \, a^3(t) \left\{ \frac{1}{2} \dot{\vec{\Phi}}^2(\vec{x}, t) - \frac{1}{2 a(t)^2} [\vec{\nabla} \vec{\Phi}(\vec{x}, t)]^2 - V(\vec{\Phi}(\vec{x}, t)) \right\} \quad (2.4)$$

$$V(\Phi) = \frac{1}{2} [-m_0^2 + \xi_0 \mathcal{R}] \vec{\Phi}^2 + \frac{\lambda_0}{8N} (\vec{\Phi} \cdot \vec{\Phi})^2 + \frac{m_0^4}{2\lambda} \quad (2.5)$$

$$\mathcal{R} = 6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \quad (2.6)$$

with \mathcal{R} the Ricci scalar and ξ_0 has been introduced with the purpose of renormalization. Since the time evolution of the quantum density matrix is determined by Liouville's equation, we need the Hamiltonian which requires the canonical momentum conjugate to $\Phi(\vec{x}, t)$

$$\vec{\Pi}(\vec{x}, t) = a^3(t) \dot{\vec{\Phi}}(\vec{x}, t) \quad (2.7)$$

The Hamiltonian becomes

$$H(t) = \int d^3x \left\{ \frac{\vec{\Pi}^2(\vec{x}, t)}{2 a^3(t)} + \frac{a(t)}{2} (\vec{\nabla} \vec{\Phi})^2 + a^3(t) V(\vec{\Phi}) \right\} \quad (2.8)$$

In the Schrödinger representation (at an arbitrary time t), the canonical momentum is represented as

$$\Pi^a(\vec{x}) = -i \frac{\delta}{\delta \Phi^a(\vec{x})} \quad ; a = 1, \dots, N$$

and the functional density matrix $\hat{\rho}$ with matrix elements in the Schrödinger representation $\rho[\Phi^a(\vec{\cdot}), \tilde{\Phi}^b(\vec{\cdot}); t]$ obeys the Liouville equation, which in the Schrödinger representation becomes the functional differential equation [12]

$$i \frac{\partial}{\partial t} \rho[\Phi^a(\vec{\cdot}), \tilde{\Phi}^b(\vec{\cdot}); t] = \left(H \left[\frac{\delta}{\delta \Phi(\vec{\cdot})}; \vec{\Phi} \right] - H \left[\frac{\delta}{\delta \tilde{\Phi}(\vec{\cdot})}; \tilde{\vec{\Phi}} \right] \right) \rho[\Phi^a(\vec{\cdot}), \tilde{\Phi}^b(\vec{\cdot}); t] \quad (2.9)$$

To leading order the large N limit can be implemented by the following Hartree-like factorization [18,14] (for an alternative formulation of the large N limit see [18])

$$(\vec{\Phi} \cdot \vec{\Phi})^2 \rightarrow 2 \langle \vec{\Phi} \cdot \vec{\Phi} \rangle \vec{\Phi} \cdot \vec{\Phi} - \langle \vec{\Phi} \cdot \vec{\Phi} \rangle^2 \quad ; \quad \langle \vec{\Phi} \cdot \vec{\Phi} \rangle = N \langle \Phi^a \Phi^a \rangle \text{ (no sum over } a) \quad (2.10)$$

where the expectation value is in the time evolved density matrix $\rho(t)$ which is the solution of the Liouville equation above.

It is convenient to introduce the spatial Fourier transform of the fields as

$$\vec{\Phi}(\vec{x}, t) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{k}} \vec{\Phi}_{\vec{k}}(t) e^{i\vec{k} \cdot \vec{x}} \quad (2.11)$$

with Ω the spatial volume, and a similar expansion for the canonical momentum $\vec{\Pi}(\vec{x}, t)$. The Hamiltonian becomes

$$H = \sum_{\vec{k}} \left\{ \frac{1}{2 a^3(t)} \vec{\Pi}_{\vec{k}} \cdot \vec{\Pi}_{-\vec{k}} + \frac{a^3(t)}{2} W_k^2(t) \vec{\Phi}_{\vec{k}} \cdot \vec{\Phi}_{-\vec{k}} - a^3(t) \frac{\lambda}{8N} \langle \vec{\Phi} \cdot \vec{\Phi} \rangle^2 + \frac{m_0^4}{2\lambda} \right\} \quad (2.12)$$

$$W_k^2(t) = -m_0^2 + \xi_0 \mathcal{R} + \frac{k^2}{a^2(t)} + \frac{\lambda}{2} \int \frac{d^3k}{(2\pi)^3} \langle \Phi_k^a \Phi_{-\vec{k}}^a \rangle(t) \text{ (no sum over } a) \quad (2.13)$$

This Hamiltonian describes a set of infinitely many harmonic oscillators, that are only coupled through the self-consistent condition in the frequencies (2.13). Since the effective Hamiltonian is quadratic in terms of the self-consistent frequencies, we propose the following Gaussian Ansatz for the functional density matrix elements in the Schrödinger representation [12]

$$\rho[\vec{\Phi}, \vec{\tilde{\Phi}}, t] = \prod_{\vec{k}} \mathcal{N}_k(t) \exp \left\{ -\frac{A_k(t)}{2} \vec{\Phi}_k \cdot \vec{\Phi}_{-k} - \frac{A_k^*(t)}{2} \vec{\tilde{\Phi}}_k \cdot \vec{\tilde{\Phi}}_{-k} - B_k(t) \vec{\Phi}_k \cdot \vec{\tilde{\Phi}}_{-k} \right\} \quad (2.14)$$

This form of the density matrix is dictated by the hermiticity condition $\rho^\dagger[\vec{\tilde{\Phi}}, \vec{\Phi}, t] = \rho^*[\vec{\Phi}, \vec{\tilde{\Phi}}, t]$; as a result of this condition $B_k(t)$ is real. The kernel $B_k(t)$ determines the amount of mixing in the density matrix. If $B_k = 0$, the density matrix corresponds to a pure state because it is a direct product of a state vector times its adjoint conjugate.

The Liouville equation (2.9) becomes [12]

$$\begin{aligned} i \frac{\partial}{\partial t} \rho[\vec{\Phi}, \vec{\tilde{\Phi}}, t] = \sum_{\vec{k}} \left\{ -\frac{1}{2a^3(t)} \left(\frac{\delta^2}{\delta \vec{\Phi}_{\vec{k}} \cdot \delta \vec{\Phi}_{-\vec{k}}} - \frac{\delta^2}{\delta \vec{\tilde{\Phi}}_{\vec{k}} \delta \vec{\tilde{\Phi}}_{-\vec{k}}} \right) \right. \\ \left. + \frac{a^3(t)}{2} W_k^2(t) \left(\vec{\Phi}_{\vec{k}} \cdot \vec{\Phi}_{-\vec{k}} - \vec{\tilde{\Phi}}_{\vec{k}} \cdot \vec{\tilde{\Phi}}_{-\vec{k}} \right) \right\} \rho[\vec{\Phi}, \vec{\tilde{\Phi}}, t] \end{aligned} \quad (2.15)$$

The equations for the kernels in the density matrix (A_k, B_k) are obtained by comparing similar powers of $\vec{\Phi}_{\vec{k}}$ on both sides of the above equation. We obtain the following equations for the covariances:

$$i \frac{\dot{\mathcal{N}}_k}{\mathcal{N}_k} = \frac{1}{2a^3(t)} (A_k - A_k^*) \quad (2.16)$$

$$i \dot{A}_k = \frac{A_k^2 - B_k^2}{a(t)^3} - a^3(t) W_k^2(t) \quad (2.17)$$

$$i \dot{B}_k = \frac{B_k}{a^3(t)} (A_k - A_k^*) \quad (2.18)$$

The equation for $B_k(t)$ reflects the fact that an initial pure state with $B_k(0) = 0$ remains pure under time evolution. From these equations it becomes clear that the only independent equation is that for $A_k(t)$. In particular, writing $A_k(t) = A_{R,k}(t) + iA_{I,k}(t)$ we find that the ratios $A_{R,k}(t)/B_k(t)$ and $\mathcal{N}_k(t)/[A_{R,k}(t) + B_k(t)]^{\frac{1}{2}}$ are constant in time, the latter being a consequence of unitary time evolution of the density matrix. Exploiting the proportionality between $A_{R,k}(t)$ and $B_k(t)$ we introduce a new variable [12]

$$\mathcal{A}_k(t) = \mathcal{A}_{R,k}(t) + i\mathcal{A}_{I,k}(t)$$

by defining

$$A_{R,k}(t) = \mathcal{A}_{R,k}(t) \coth \Theta_k \quad (2.19)$$

$$B_k(t) = -\frac{\mathcal{A}_{R,k}(t)}{\sinh \Theta_k} \quad (2.20)$$

$$A_{I,k}(t) = \mathcal{A}_{I,k}(t) \quad (2.21)$$

This new variable obeys the following Ricatti equation of motion

$$i\dot{\mathcal{A}}_k(t) = \frac{\mathcal{A}_k^2(t)}{a^3(t)} - a^3(t) W_k^2(t) \quad (2.22)$$

This equation can be linearized by defining

$$\mathcal{A}_k(t) = -ia^3(t) \frac{\dot{\varphi}_k^*(t)}{\varphi_k^*(t)} \quad (2.23)$$

and the mode functions $\varphi_k(t)$ obey the following equation of motion

$$\ddot{\varphi}_k(t) + 3 \frac{\dot{a}(t)}{a(t)} \dot{\varphi}_k(t) + W_k^2(t) \varphi_k(t) = 0 \quad (2.24)$$

The equations of motion (2.24) are recognized as the *Heisenberg* equations of motion for $\vec{\Phi}_k(t)$ obtained from the Hamiltonian (2.12). This observation leads to a rather clear interpretation of the mode functions $\varphi_k(t)$ as a basis for the expansion of the Heisenberg field operator $\vec{\Phi}_{\vec{k}}(t)$, i.e.

$$\vec{\Phi}_{\vec{k}}(t) = \frac{1}{\sqrt{2}} \left[\vec{\mathbf{a}}_{\vec{k}} \varphi_k(t) + \vec{\mathbf{a}}_{-\vec{k}}^\dagger \varphi_k^*(t) \right] \quad (2.25)$$

where the annihilation and creation operators $\vec{\mathbf{a}}_{\vec{k}} ; \vec{\mathbf{a}}_{-\vec{k}}^\dagger$ are independent of time in the Heisenberg picture and define a Fock representation. We will normalize the mode functions $\varphi_k(t)$ so that the Wronskian is given by

$$\mathcal{W}[\varphi_k(t), \varphi_k^*(t)] = \varphi_k^*(t) \dot{\varphi}_k(t) - \varphi_k(t) \dot{\varphi}_k^*(t) = -2i \left[\frac{a(t_0)}{a(t)} \right]^3 \quad (2.26)$$

where t_0 is the initial time at which the density matrix is prepared. Hence we find

$$\langle \Phi_{\vec{k}}^a \Phi_{-\vec{k}}^a \rangle(t) = \frac{|\varphi_k(t)|^2}{2a^3(t_0)} \coth \frac{\Theta_k}{2} \quad , \quad \forall a = 1, \dots, N \quad (2.27)$$

Without loss of generality, we choose $a(t_0) = 1$ which can always be done by a simple rescaling of lengths. Furthermore, we will choose Θ_k to reflect a thermal density matrix at the time t_0 in terms of the self-consistent frequencies which are discussed below.

A. Conformal time analysis, initial conditions and renormalization

It is convenient to change variables to conformal time η defined as

$$\eta = \int_{t_0}^t \frac{dt'}{a(t')} \quad ; \quad \eta(t = t_0) = 0 \quad , \quad (2.28)$$

which is chosen to vanish at the initial time.

The scale factor in conformal time $C(\eta)$ and conformally rescaled mode functions $f_k(\eta)$ are given by

$$C(\eta) = a(t(\eta)) \quad ; \quad C(0) = 1 \quad (2.29)$$

$$f_k(\eta) = a(t) \frac{\varphi_k(t)}{\sqrt{\omega_k}} \quad (2.30)$$

The conformally rescaled mode functions obey the Schrödinger-like differential equation

$$\left[\frac{d^2}{d\eta^2} + k^2 + C^2(\eta) M^2(\eta) \right] f_k(\eta) = 0 \quad (2.31)$$

$$M^2(\eta) = -m_0^2 + \left(\xi_0 - \frac{1}{6}\right) \mathcal{R}(\eta) + \frac{\lambda}{8\pi^2} \int k^2 dk \frac{|f_k(\eta)|^2}{C^2(\eta)} \omega_k \coth \frac{\Theta_k}{2} \quad (2.32)$$

$$\mathcal{R}(\eta) = 6 \frac{C''(\eta)}{C^3(\eta)} \quad (2.33)$$

where primes denote derivative with respect to conformal time. We will choose the following initial conditions on the mode functions

$$f_k(0) = \frac{1}{\omega_k} \quad ; \quad f'_k(0) = -i \quad (2.34)$$

$$\omega_k = \sqrt{k^2 + M^2(0)} \quad (2.35)$$

in this manner the mode functions at the initial time represent positive frequency (particle) modes. We now choose

$$\Theta_k = \frac{\omega_k}{T_i} \quad (2.36)$$

so that the initial density matrix at $t = t_0; \eta = 0$ describes a statistical ensemble in local thermal equilibrium with an initial temperature T_i for the conformal modes describing (quasi) particles of self-consistent frequencies ω_k . We will not try to justify this choice of initial state in local thermal equilibrium and simply assume that it provides a physically reasonable description of the state prior to the phase transition.

The renormalization aspects had already been studied in detail in references [12,13,19] with the result that the quadratic and logarithmic divergences in terms of an ultraviolet cutoff Λ can be absorbed in mass, coupling and conformal coupling renormalization $m_0^2 \rightarrow m_R^2; \lambda_0 \rightarrow \lambda_R; \xi_0 \rightarrow \xi_R$. We refer the reader to those references for details and highlight that the main result of the renormalization program is that

$$M^2(\eta) = -m_R^2 + \left(\xi_R - \frac{1}{6}\right) \mathcal{R}(\eta) + \frac{\lambda_R}{2} \langle \psi^2 \rangle_R \quad (2.37)$$

with

$$\langle \psi^2 \rangle_R = I_R + J \quad (2.38)$$

$$I_R = \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{\omega_k |f_k(\eta)|^2}{2 C^2(\eta)} - \frac{1}{2 k C^2(\eta)} + \frac{\theta(k-K)}{4k^3} \left[-\frac{\mathcal{R}}{6} - \left(\frac{C'(0)}{C(\eta)} \right)^2 - m_R^2 + \left(\xi_R - \frac{1}{6}\right) \mathcal{R}(\eta) + \frac{\lambda_R}{2} \langle \psi^2 \rangle_R \right] \right\} \quad (2.39)$$

$$J = \int \frac{d^3k}{(2\pi)^3} \frac{|f_k(\eta)|^2}{C^2(\eta)} \frac{\omega_k}{\exp \frac{\omega_k}{T_i} - 1} \quad (2.40)$$

with K an arbitrary renormalization scale [12]. The explicit relation between bare and renormalized parameters can be found in [12,13,19]. The finite temperature contribution J can be written by separating in the integral the large k behavior of the mode functions $f_k(\eta) \sim e^{-ik\eta}/\sqrt{k}$ for $k \gg C(\eta) M(\eta)$. Assuming that the initial temperature $T_i \gg M(0)$ we can separate the contribution from the large wave-vectors $k \geq T \gg M(0)$ as in the hard-thermal loop approximation in thermal field theory [20,21]. The high temperature limit of J has been obtained in reference [12], the leading contribution is obtained in the hard thermal loop limit

$$J_{HTL} = \frac{T_{eff}^2(\eta)}{12} \quad ; \quad T_{eff}(\eta) = T_i \frac{C(0)}{C(\eta)} \quad ; \quad T_{eff}(0) = T_i \quad (2.41)$$

This term is completely determined the very short wavelength modes $k \geq T \gg M(0)$ and reflects the adiabatic *cooling* of the short wavelength modes in the initial state by the cosmological expansion.

Introducing the critical temperature as

$$T_c = m_R \sqrt{\frac{24}{\lambda_R}} \quad (2.42)$$

we find that the hard thermal loop contribution J_{HTL} combines with the renormalized mass term $-m_R^2$ in $M^2(\eta)$ (2.32) to yield

$$-m_R^2 + \frac{\lambda_R}{2} J_{HTL}(\eta) = m_R^2 \left[\frac{T_{eff}^2(\eta)}{T_c^2} - 1 \right] \quad (2.43)$$

Clearly the leading order finite temperature contribution, i.e. the hard thermal loop limit gives rise to an effective η dependent mass squared term that changes sign when the effective temperature falls below the critical, thus triggering the phase transition.

There are sub-leading corrections in the high temperature limit that have been studied in detail in reference [12]. The term linear in the initial temperature corresponds to the *classical* contribution for $\omega_k \ll T_i$, i.e. $k \ll T_i$ which is given by

$$J_{cl}(\eta) = T_i \int \frac{d^3k}{(2\pi)^3} \frac{|f_k(\eta)|^2}{C^2(\eta)} \quad (2.44)$$

The contributions from $J_{HTL}(\eta)$ and $J_{cl}(\eta)$ are the most important ones in the weak coupling limit $\lambda_R \ll 1$ as can be seen as follows. The leading order correction $J_{HTL}(\eta)$ determines the effective squared mass, which is positive for $T_{eff}(\eta) > T_c$ and becomes negative for $T_{eff}(\eta) < T_c$ thus triggering the phase transition. Once the effective temperature falls below the critical, long-wavelength instabilities begin growing exponentially and the long-wavelength modes acquire non-perturbatively large amplitudes as argued in the introduction and studied explicitly in detail below. These long-wavelength modes with $k \ll T_i$ determine the contribution $J_{cl}(\eta)$, which is the dominant contribution from the infrared sector in the high temperature limit. In terms of the critical temperature (2.42) and the ratio $T_i/T_c \geq 1$ the classical contribution leads to

$$\frac{\lambda_R}{2} J_{cl}(\eta) = m_R \frac{\sqrt{24\lambda_R}}{4\pi^2} \frac{T_i}{T_c} \int_0^\kappa k^2 dk \frac{|f_k(\eta)|^2}{C^2(\eta)} \quad (2.45)$$

where κ is a cutoff that determines the maximum wavevector that will be unstable. Numerically it is found that the integral becomes insensitive to the choice of this cutoff for $\kappa \geq m_R$. The next contribution is logarithmic in temperature [12] and is of order $\lambda_R \ln(\lambda_R)$, therefore subleading in the high temperature and weak coupling limit. Furthermore, the zero temperature part is finite after renormalization and subleading in the weak coupling limit since it is of $\mathcal{O}(\lambda_R)$ as compared to the $\mathcal{O}(1)$; $\mathcal{O}(\sqrt{\lambda_R})$ for the $\mathcal{O}(T_i^2)$; $\mathcal{O}(T_i)$ contributions from J_{HTL} ; J_{cl} respectively. Hence the zero temperature part that remains after renormalization will be neglected.

We finally obtain the equations of motion in leading order in the high temperature and weak coupling expansion after the following steps

- Rescaling all dimensionful quantities, $k, \eta, f_k(\eta), \omega_k$, etc. in terms of the only dimensionful parameter m_R . Effectively this amounts to setting $m_R = 1$ and all dimensionful parameters are understood in units of m_R .
- After renormalization of couplings and mass we neglect the zero temperature contribution, which is higher order in the weak coupling λ_R and keep only the leading order terms $J_{HTL}(\eta)$ and $J_{cl}(\eta)$ neglecting higher order contributions which are subleading in the high initial temperature and weak coupling limits.
- We choose the case of minimal coupling $\xi_R = 0$.

In summary, the equations of motion for the mode functions and the initial conditions are

$$\left[\frac{d^2}{d\eta^2} + q^2 + C^2(\eta)M^2(\eta) \right] f_q(\eta) = 0 \quad (2.46)$$

$$M^2(\eta) = \frac{T_{eff}^2(\eta)}{T_c^2} - 1 - \frac{C''(\eta)}{C^3(\eta)} + g\Sigma(\eta) \quad (2.47)$$

$$\Sigma(\eta) = \int_0^1 q^2 dq \frac{|f_q(\eta)|^2}{C^2(\eta)} \quad ; \quad g = \frac{\sqrt{24\lambda_R}}{4\pi^2} \frac{T_i}{T_c} \quad (2.48)$$

$$f_q(0) = \frac{1}{\omega_q} \quad ; \quad f'_q(0) = -i\omega_q f_q(0) \quad (2.49)$$

$$\omega_q = \sqrt{q^2 + \frac{T_i^2}{T_c^2} - 1} \quad (2.50)$$

where we have set the cutoff $\kappa = m_R \equiv 1$ which will be justified numerically below, and neglected perturbatively small corrections of $\mathcal{O}(\sqrt{\lambda_R})$ in ω_q . The term $g\Sigma(\eta)$ describes the back-reaction of the quantum and thermal fluctuations.

With the background energy density determined by the combination of matter and radiation as given in eq. (2.3) a simple expression is obtained for $C(\eta)$ by introducing the quantities

$$r_i = \frac{\rho_M^i}{\rho_R^i} \quad ; \quad \frac{H_i^2}{1 + r_i} = \frac{8\pi\rho_R^i}{3M_{Pl}^2} \quad (2.51)$$

where we have used $a(t_0) = 1$. A straightforward integration leads to

$$C(\eta) = 1 + H_i \eta + \frac{H_i^2 r_i}{4(1+r_i)} \eta^2 = \begin{cases} 1 + H_i \eta & \text{for Radiation dominated} \\ 1 + H_i \eta + \frac{H_i^2}{4} \eta^2 & \text{for Matter dominated} \end{cases} \quad (2.52)$$

Rescaling H_i and η in terms of m_R we choose the value $H_i = 1/2$ for convenience. This value is motivated by a choice of an initial radiation energy density $\sim (10^{16})^4 \text{ (Gev)}^4$ with a mass scale $\sim 10^{12} \text{ Gev}$. Other values of H_i lead to a proper re-scaling of the time scales and therefore a quantitative change in the results, but the qualitative features described below are robust.

III. DYNAMICS OF SYMMETRY BREAKING:

We begin our study of the dynamics of symmetry breaking by focusing on the case of a radiation dominated FRW cosmology, i.e. we set $r = 0$ in (2.52) and as explained above we have chosen $H_i = 1/2$ for convenience. The dynamics is completely determined by the set of equations (2.46-2.50) with $C(\eta) = 1 + \eta/2$.

Before embarking on a numerical study of these equations we can obtain insight into the dynamics by neglecting the backreaction term $g\Sigma(\eta)$ in (2.47). This approximation is valid in the weak coupling limit and for early times when $g\Sigma(\eta) \ll 1$. During the time scale for which $T_{eff}(\eta) > T_c$ the effective squared mass is positive and the mode functions are oscillatory functions of conformal time. When the effective temperature falls below the critical, the effective squared mass becomes negative and long-wavelength modes with momentum $q^2 < C^2(\eta) M^2(\eta)$ become unstable and begin to grow almost exponentially, obviously the growth rate is the largest for the smaller q . We can obtain an estimate of the behavior of the long-wavelength modes after the temperature falls below the critical via a WKB expansion which after approximating $M^2(\eta) \approx -1 - C''(\eta)/C^3(\eta)$ leads to

$$f_{q \sim 0}(\eta) \sim \frac{e^{\int^\eta C(\eta') |M(\eta')| d\eta'}}{\sqrt{C(\eta) |M(\eta)|}} \quad (3.1)$$

which reflect an exponential growth of long-wavelength fluctuations as a result of *linear instabilities* of the long-wavelength modes below the critical temperature. This behavior in turn entails that the backreaction term also grows exponentially as

$$g\Sigma(\eta) \sim g \frac{e^{2 \int^\eta C(\eta') |M(\eta')| d\eta'}}{C^3(\eta) |M(\eta)|} \quad (3.2)$$

and becomes non-perturbatively large at a time scale η_{NL} when it begins to compete with the tree level term, i.e. when

$$g\Sigma(\eta_{NL}) \sim 1 \quad (3.3)$$

The condition (3.3) determines the time scale η_{NL} at which the non-linearities become important and must be treated non-perturbatively. This condition has a simple physical interpretation: it implies that the amplitude of the long wavelength fluctuations probe the broken symmetry states, i.e. since

$$g\Sigma(\eta) \approx \frac{\lambda_R}{2} \langle \Phi^a \Phi^a \rangle_R$$

restoring m_R as the scale unit we see that

$$g\Sigma(\eta_{NL}) \approx m_R^2 \Rightarrow \langle \Phi^a \Phi^a \rangle_R \approx 2 m_R^2 / \lambda_R \sim \Phi_0^2$$

with Φ_0^2 being the minimum of the renormalized tree level potential. Once $g\Sigma \approx m_R^2$ the backreaction compensates for the negative squared mass and the long-wavelength instabilities shut-off. This picture is confirmed by a detailed numerical analysis of the equations of motion (2.46)-(2.50) with $C(\eta) = 1 + \eta/2$ for a radiation dominated FRW cosmology ($r = 0$).

Fig. 1 displays the effective squared mass $M^2(\eta)$ given by eq. (2.47) for a radiation dominated ($r = 0$) cosmology with the value of the parameters $T_i/T_c = 3$; $g = 10^{-5}$; $H_i = 1/2$ and all dimensionful quantities had been rescaled by m_R . The effective squared mass begins with a positive value at $\eta = 0$ since $T_i > T_c$ but diminishes as a consequence of the adiabatic cooling through the cosmological expansion. The back reaction $g\Sigma(\eta)$ is displayed in Fig. 2, it is initially perturbatively small during the stage in which $T_{eff}(\eta) > T_c$. Once the effective temperature falls below the critical, long-wavelength fluctuations begin to grow approximately as in eq.(3.1) as a result of the linear instabilities and the back reaction term begins to grow exponentially and to compete with the tree level term. The rapid exponential growth of the back-reaction terms results in an overshooting and the effective mass $M^2(\eta)$ becomes positive. When this happens the mode functions become again oscillatory and their amplitude diminishes causing a damped oscillatory behavior in $g\Sigma(\eta)$ and consequently in $M^2(\eta)$. However, in the Schrödinger-like equations for the mode functions (2.46) the mass term is $C^2(\eta)M^2(\eta)$, thus while $M^2(\eta)$ becomes positive and oscillatory and eventually damps out by dephasing of the oscillations the prefactor $C^2(\eta)$ makes this effective mass to grow during some period of time. The effective mass $C^2(\eta)M^2(\eta)$ for the mode functions $f_q(\eta)$ is displayed in Fig. 3 which clearly shows the initial cooling, the growth of the backreaction and the competition between the damping of the oscillations in $M^2(\eta)$ and the growth of the scale factor. This figure also reveals the striking asymptotic behavior that $C^2(\eta)M^2(\eta) \rightarrow 0$ as $\eta \rightarrow \infty$. A detailed numerical analysis reveals that asymptotically at long time $C^2(\eta)M^2(\eta) \rightarrow \mathcal{O}(1/\eta^2)$, a result that will be understood analytically below. The combination of figures (1-3) reveals three different time scales after the phase transition, i.e. when $T_{eff} < T_c$

- Early time ($\eta < \eta_{NL}$): this stage is dominated by linear instabilities that result in the exponential growth of long-wavelength modes. The back-reaction contribution $g\Sigma(\eta)$ grows exponentially but remains perturbatively small.
- Intermediate time ($\eta \sim \eta_{NL}$): during this stage the back-reaction begins to be comparable to the tree level contribution to the mass, i.e. $g\Sigma(\eta) \sim m_R^2$, the back-reaction begins to shut-off the linear instabilities but overshoots resulting in an oscillatory effective mass. This stage determines the onset of non-linear evolution since the back-reaction is of the same order as the tree level term. The damping of $M^2(\tau)$ competes with the growth of the scale factor $C^2(\eta)$ resulting in that the effective squared mass $C^2(\eta)M^2(\eta)$ still grows.
- Asymptotic regime ($\eta \gg \eta_{NL}$): This regime is fully non-linear and the detailed numerical analysis reveals that the effective mass $C^2(\eta)M^2(\eta)$ *vanishes* asymptotically. Numerically we find that in this regime

$$C^2(\eta)M^2(\eta) \stackrel{\eta \gg 1}{\approx} -\frac{(2p+3)(2p+1)}{4\eta^2} \quad \text{for } C(\eta) = \left(\frac{\eta}{2}\right)^p \quad (3.4)$$

with $p = 1, 2$ for radiation and matter dominated respectively. The result (3.4) implies a very delicate mechanism of cancellation between the back-reaction and tree level terms, which will be understood analytically below.

A. Asymptotic dynamics and emergence of scaling:

From the expression for $M^2(\eta)$ given by eq. (2.47) the vanishing of $C^2(\eta)M^2(\eta)$ in the asymptotic region, when $T_{eff}(\eta) \ll T_c$ leads to the following sum rule in the asymptotic regime $\eta > \eta_{NL} \gg 1$ (in units of m_R)

$$g \int_0^\infty q^2 dq |f_q(\eta)|^2 = C^2(\eta) \stackrel{\eta \gg 1}{\approx} \left(\frac{\eta}{2}\right)^{2p} \quad (3.5)$$

where $p = 1$ for radiation dominated and $p = 2$ for matter dominated.

The constraint that the integral on the left hand side must lead to a power law suggests the following *scaling ansatz* for the mode functions

$$|f_q(\eta)|^2 = \eta^\alpha |\mathcal{F}(x)|^2 \quad ; \quad x = q\eta \quad (3.6)$$

Comparing the powers and the coefficients leads to the following constraints

$$\alpha = 2p + 3 \quad (3.7)$$

$$g \int_0^\infty x^2 dx |\mathcal{F}(x)|^2 = 2^{-2p} \quad (3.8)$$

Assuming that the scaling function $\mathcal{F}(x)$ is regular at $x = 0$, the scaling *ansatz* (3.6) leads to a remarkable conclusion: the $q = 0$ mode function is of the form

$$f_0(\eta) = \eta^{\frac{\alpha}{2}} \mathcal{F}(0) \quad (3.9)$$

leading to the result

$$C^2(\eta)M^2(\eta) = -\frac{f_0''(\eta)}{f_0(\eta)} = -\frac{(2p+3)(2p+1)}{4\eta^2} = \begin{cases} -15/4\eta^2 & \text{for Radiation dominated} \\ -35/4\eta^2 & \text{for Matter dominated} \end{cases} \quad (3.10)$$

which is precisely the numerical result (3.4). The result (3.10) in turn leads to the following equations of motion for the $q \neq 0$ mode functions

$$\left[\frac{d^2}{d\eta^2} + q^2 + \frac{(2p+3)(2p+1)}{4\eta^2} \right] f_q(\eta) = 0 \quad (3.11)$$

with the general solutions in terms of the scaling variable $x = q\eta$

$$f_q(\eta) = A_q \eta^{\frac{5}{2}} \frac{J_2(x)}{x^2} + B_q \frac{x^2}{\eta^{\frac{3}{2}}} N_2(x) \quad \text{for Radiation dominated} \quad (3.12)$$

$$f_q(\eta) = A_q \eta^{\frac{7}{2}} \frac{J_3(x)}{x^3} + B_q \frac{x^3}{\eta^{\frac{5}{2}}} N_3(x) \quad \text{for Matter dominated} \quad (3.13)$$

Where A_q , B_q are numerical constants and $J_{p+1}(x)$; $N_{p+1}(x)$ are Bessel and Neumann functions respectively. The constant Wronskian between $f_q(\eta)$ and its complex conjugate determines that neither A_q nor B_q could vanish. However in the asymptotic limit $\eta \rightarrow \infty$ and at *fixed* x , two important simplifications occur: i) the term proportional to $N_{p+1}(x)$ becomes subleading and ii) $q = x/\eta \rightarrow 0$ and $A_q \rightarrow A_0$, thus in the asymptotic regime for very large η and *fixed* x we find that the asymptotic behavior of the mode functions is dominated by the $J_{p+1}(x)$ contributions with a constant coefficient A_0 , i.e. asymptotically for $\eta \gg 1$, x fixed, the leading contribution to the solutions are of the *scaling form*

$$f_q(\eta) \stackrel{\eta \gg 1}{\simeq} A_0 \eta^{\frac{5}{2}} \frac{J_2(x)}{x^2} + \mathcal{O}\left(\frac{1}{\eta^{\frac{3}{2}}}\right) \quad \text{for Radiation dominated} \quad (3.14)$$

$$f_q(\eta) = A_0 \eta^{\frac{7}{2}} \frac{J_3(x)}{x^3} + \mathcal{O}\left(\frac{1}{\eta^{\frac{5}{2}}}\right) \quad \text{for Matter dominated} \quad (3.15)$$

In this asymptotic region, the absolute value of the coefficient A_0 is completely determined by the constraint (3.8) resulting from the sum rule (3.5) and the integral [24]

$$\int_0^\infty \frac{dx}{x^{2p}} [J_{p+1}(x)]^2 = \frac{\sqrt{\pi} \Gamma(2p)}{2^{2p+1} \Gamma(2p + \frac{3}{2}) \Gamma(p + \frac{1}{2})^2}.$$

We find,

$$|A_0|^2 = \begin{cases} \frac{15\pi}{16g} = 2.94524 \dots /g & \text{for Radiation dominated} \\ \frac{2835\pi}{512g} = 17.39534 \dots /g & \text{for Matter dominated} \end{cases} \quad (3.16)$$

Figs. 5-7 display $g\eta^{-5} |f_q(\eta = 240)|^2$; $g\eta^{-5} |f_q(\eta = 400)|^2$ and $g|A_0 J_2(x)/x^2|^2$ vs. x respectively for a radiation dominated cosmology with $g = 10^{-5}$; $T_i/T_c = 3$. These figures are indistinguishable from each other, furthermore we have checked numerically that $\eta^{-\frac{5}{2}} f_{q=0}(\eta)$ approaches a constant asymptotically both for the real and the imaginary part in a radiation dominated FRW cosmology. Similar results had been obtained numerically for the case of matter domination, in particular confirming the scaling behavior (3.15) with the appropriate coefficient given by (3.16). In this case also both the real and imaginary part of $\eta^{-\frac{7}{2}} f_{q=0}(\eta)$ approach a constant asymptotically and the effective mass in both cases oscillates with small amplitude around a mean value given by (3.10) as is shown explicitly in fig. (8).

Thus a detailed numerical integration of the equations of motion confirms the scaling *ansatz* and leads to the following conclusions:

- Asymptotically the effective mass $C^2(\eta)M^2(\eta) \stackrel{\eta \gg 1}{\simeq} -\frac{(2p+3)(2p+1)}{4\eta^2}$. The vanishing of the effective mass leads to the sum rule (3.5) which suggests the scaling *ansatz* (3.6) with the power law given by (3.7). The self-consistent solution in the asymptotic regime is given by (3.12) for radiation dominated and (3.13) for matter dominated. Neither of

the coefficients $A_q; B_q$ vanishes as a consequence of the constancy of the Wronskian of f_q and its complex conjugate. However, the asymptotic regime $\eta \gg 1$, and x fixed is completely determined by the scaling form of the solutions given by eqs. (3.14), (3.15) with the modulus squared of the coefficient A_0 determined by the sum rule and given by (3.16).

- In the asymptotic regime $\eta \gg 1$, x fixed which is dominated by the scaling solution we see that only the long wavelength modes are *relevant*, i.e. $q = x/\eta \ll 1$ (in units of m_R), justifying keeping only the classical part J_{cl} in eq. (2.44) and using the cutoff $\kappa \sim m_R$ in the integral in (2.45).
- The amplitude of the long wavelength modes $q = x/\eta$ that dominate the scaling regime become *non-perturbatively large* as can be seen in figs. (5,6) and is a consequence of the sum rule (3.8) that fixes the amplitude of the coefficient of the scaling solution. The fact that the long-wavelength modes become non-perturbatively large and dominate the dynamics is a consequence of the early time *linear instabilities* that result in an exponential growth of these modes.
- A noteworthy feature of the scaling solution is that its phase has *frozen* i.e. it became time independent and completely determined by the phase of A_0 . We note that if the two linearly independent solutions in eq. (3.12,3.13) had the same amplitude the phase of the mode functions will be time and momentum dependent. The freezing of the phase and the non-perturbative large amplitude of the long-wavelength modes entail that these mode functions that originally had *quantum* initial conditions (as can be seen by restoring \hbar to obtain the quantum commutators) had become *classical*.

B. Scaling corrections

Although the leading asymptotic behavior of the mode functions is determined by the scaling forms (3.14, 3.15) the subleading contributions determined by the terms containing the Neumann functions $N_{p+1}(x)$ in eqs. (3.12, 3.12) are important corrections to scaling behavior and are required for the self-consistency of the solutions.

The analytic and numerical result $C^2(\eta)M^2(\eta) \stackrel{\eta \gg 1}{\approx} -\frac{(2p+3)(2p+1)}{4\eta^2}$ entails *two* very stringent constraints. The first one leads to the cancellation between the tree level contribution and the backreaction term $g\Sigma(\eta) = 1$ to leading order in $C(\eta)$ and leads to the sum rule eq. (3.5). However the fact that $C^2(\eta)M^2(\eta) \stackrel{\eta \gg 1}{\approx} -\frac{(2p+3)(2p+1)}{4\eta^2}$ in the asymptotic region when $T_{eff}(\eta)/T_c \ll 1$ implies that

$$C^2(\eta)M^2(\eta) = g \int_0^1 q^2 dq |f_q(\eta)|^2 - C^2(\eta) \stackrel{\eta \gg 1}{\approx} -\frac{(2p+3)(2p+1)}{4\eta^2} \quad (3.17)$$

Inserting the solutions (3.12)-(3.13) in eq. (3.17), changing integration variables to $q = x/\eta$ and taking the asymptotic limit $\eta \gg 1$, x fixed inside the integral it is seen that the leading contribution arises from the terms proportional to $J_{p+1}(x)$ which cancel $C^2(\eta)$. The next order term arises from the crossed term between the contribution proportional to the $J_{p+1}(x)$ and that from the $N_{p+1}(x)$ which leads to another sum rule

$$\frac{2g \operatorname{Re}[A_0 B_0^*]}{\eta^2} \int_0^\infty x^2 dx J_{p+1}(x) N_{p+1}(x) = -\frac{(2p+3)(2p+1)}{4\eta^2} \quad (3.18)$$

which describes the asymptotic behavior of $C^2(\eta)M^2(\eta)$.

Using the result [24]

$$\int_0^\infty x^2 dx J_{p+1}(x) N_{p+1}(x) = -\frac{(2p+3)(2p+1)}{16}$$

we find that the sum-rule (3.18) constrains the value of $\operatorname{Re}[A_0 B_0^*]$ as follows,

$$\operatorname{Re}[A_0 B_0^*] = -\frac{2}{g}$$

thus we are led to the conclusion that the self-consistency condition indeed requires a *non-vanishing correction to scaling*.

Eq. (3.18) reveals that $B_0 \propto 1/\sqrt{g}$ therefore of the same order as A_0 . Thus we conclude that although the asymptotic behavior of the mode functions is dominated by the scaling form, the corrections to scaling embodied in the subleading contributions are very important for the self-consistency of the solution, they have non-perturbative amplitudes of $\mathcal{O}(1/\sqrt{g})$ and as argued above the coefficients B_q do not vanish because of the Wronskian condition on the mode functions.

C. Classicality

We argued above that the long-wavelength modes become classical in the sense that their amplitudes become non-perturbatively large $\mathcal{O}(1/\sqrt{g})$ and their phases freeze out. An important bonus of studying the dynamics in terms of a quantum density matrix is that the classicalization of long-wavelength fluctuations can be quantified in terms of the probability density (functional) in field space. Just as in quantum mechanics the probability density is the diagonal density matrix element in the Schrödinger representation, i.e.

$$\mathcal{P}(\Phi) = \rho[\Phi, \Phi, t] = \prod_{\vec{k}} \mathcal{N}_k(t) \exp \left\{ -[A_{k,R}(t) + B_k(t)] \vec{\Phi}_k \cdot \vec{\Phi}_{-k} \right\} \quad (3.19)$$

For long-wavelength modes $k \ll T_i$ we can approximate $2 \tanh \left[\frac{\omega_k}{2T_i} \right] \approx \omega_k/T_i$ and in terms of conformal time and the mode functions that obey the equations (2.46)-(2.50) we find

$$\mathcal{P}(\Phi, \eta) = \prod_{\vec{k}} \mathcal{N}_k(\eta) \exp \left\{ -\frac{C^2(\eta) \vec{\Phi}_k \cdot \vec{\Phi}_{-k}}{T_i |f_k(\eta)|^2} \right\} \quad (3.20)$$

Thus at any given fixed (conformal) time, configurations with amplitude $\Phi_k^a \sim \sqrt{T_i} |f_k(\eta)|/C(\eta)$ (in units of m_R) have a probability $\sim \mathcal{O}(1)$ of being represented in the statistical ensemble. Since $T_i \geq T_c \approx 1/g$ and in the asymptotic regime ($\eta \gg 1$; x fixed) $|f_k(\eta)| \propto 1/\sqrt{g}$, these are *large amplitude* $\mathcal{O}(1/g)$, *long wavelength* $k \leq 1/\eta$ field configurations.

Thus whereas asymptotically for $\eta \gg \eta_{NL}$ these large amplitud long-wavelength configurations are represented in the ensemble with probability $\mathcal{O}(1)$, in the initial density matrix with $|f_k(0)| \approx 1$ these field configurations are represented in the ensemble with probability $\propto e^{-\frac{1}{g}} \ll 1$ in the weak coupling limit. Under time evolution the Gaussian probability density (functional) for long-wavelength modes spreads out and large amplitude long-wavelength configurations acquire non-vanishing probabilities of being represented in the statistical ensemble. Thus in the asymptotic scaling regime, a typical field configuration found in the statistical ensemble will have amplitude $\mathcal{O}(1/g)$ and the Fourier transform of its spatial profile will be dominated by long-wavelength modes $k \ll 1/\eta$. This observation leads to a *semiclassical stochastic* description in terms of semiclassical field configurations that describe a typical member of the ensemble for $\eta \gg \eta_{NL}$

$$\Phi_{typ}^a(\vec{x}, \eta) \approx \sum_{\vec{k}} \frac{\sqrt{T_i} |f_k(\eta)|}{C(\eta)} \cos[\vec{k} \cdot \vec{x} + \delta_k^a] \quad (3.21)$$

where the phases δ_k^a are stochastic with a Gaussian distribution in order to reproduce the field correlation function obtained from the Gaussian quantum density matrix given by the integrand in (2.44). The phases δ_k represent the phases of the coefficients A_q in the scaling solutions (3.14)-(3.15) which as argued above is time independent. We note that a spatial translation can be absorbed into a redefinition of the phase δ_k and the stochastic nature of this variable in terms of a Gaussian probability distribution restores translational invariance in the ensemble averages of the semiclassical but stochastic field configurations.

IV. CONSEQUENCES OF SCALING: DYNAMICAL CORRELATION LENGTH AND EQUATION OF STATE

A. Dynamical correlation length

As described in the introduction in known systems the dynamics of phase ordering leads to the emergence of a dynamical correlation length $\xi(t)$ which determines the size of the correlated regions [6]- [9]. This dynamical correlation length plays the same role in the description of the dynamics as the static correlation length does in *static* critical phenomena. When the correlation length becomes much larger than the typical microscopic length scale in the system it becomes the only *relevant* length scale and much in the same way as in static critical phenomena a scaling regime emerges where the correlation length provides the natural scale for all dimensionful quantities. The emergence of the dynamical correlation length in dynamical critical phenomena is revealed by the equal time correlation function $\langle \Phi^a(\vec{x}, t) \Phi^b(\vec{0}, t) \rangle$. From the discussion in the previous sections, this correlation function will be dominated by the long-wavelength modes, hence we use the high temperature limit $T_i \gg k$ to find

$$\langle \Phi^a(\vec{x}, t) \Phi^b(\vec{0}, t) \rangle = \delta^{a,b} T_i \int \frac{k dk}{4\pi^2} \frac{\sin kr}{r} \frac{|f_k(\eta)|^2}{C^2(\eta)} \quad (4.1)$$

In the scaling regime when the mode functions are of the form (3.6) and multiplying by the coupling constant λ to write the result in a more familiar manner we find

$$\lambda \langle \Phi^a(\vec{x}, t) \Phi^b(\vec{0}, t) \rangle = \delta^{a,b} g D(z)$$

$$D(z) = \frac{1}{2z} \int_0^\infty x dx \sin 2xz |\mathcal{F}(x)|^2 \quad ; \quad z = \frac{r}{2\eta} \quad (4.2)$$

where we have introduced the *scaling ratio* $z = r/2\eta$. Figures 9, 10 show $gzD(z)$ as a function of z for $\eta = 240, 400$ (in units of m_R) respectively for radiation dominated cosmology obtained from the integration of the mode functions $f_k(\eta)$, and figure 11 shows $gzD(z)$ for the matter dominated case for $\eta = 400$. A remarkable feature of the correlation function $D(z)$ is that it *becomes of the order g* for $r > 2\eta$ (i.e, $z > 1$) in both cases, obviously as a result of causality. This result leads to the conclusion that the dynamics of phase ordering is described in the scaling regime as the growth of correlated regions of comoving size 2η , the reason for the factor 2 is that one edge of this region is localized at the point $\vec{0}$ and the other at \vec{r} and the boundary of this correlated region recedes at the speed of light.

Hence we obtain that the *comoving* dynamical length scale that determines the typical (comoving) size of a correlated domain is given by

$$\xi_{com}(\eta) = \eta \quad (4.3)$$

The *physical* dynamical correlation length is given by

$$\xi_{phys}(\eta) = C(\eta) \eta = d_H(\eta) \quad (4.4)$$

with $d_H(\eta)$ the size of the causal horizon. Hence we conclude that the dynamical correlation length is exactly the size of the causal horizon and within one horizon there is exactly one correlated region within which $\langle \vec{\Phi} \cdot \vec{\Phi} \rangle \approx m_R^2/\lambda_R$ i.e, the mean square root fluctuation of the field is probing the broken symmetry ground state. This result has been obtained via the full numerical evolution of the mode functions but can be understood analytically as a consequence of the scaling solutions (3.14, 3.15) that dominate the asymptotic regime for long-wavelengths. Replacing $\mathcal{F}(x)$ in (4.2) by the scaling form of the solutions given by (3.14)-(3.15) in the radiation and matter dominated cases we find

$$g z D(z) = \begin{cases} (1 - z^2)^3 F[\frac{1}{2}, \frac{5}{2}; 4; 1 - z^2] \Theta(1 - z) & \text{for Radiation dominated} \\ (1 - z^2)^5 F[\frac{1}{2}, \frac{7}{2}; 6; 1 - z^2] \Theta(1 - z) & \text{for Matter dominated} \end{cases} \quad (4.5)$$

where $F[a, b; c; z]$ is the hypergeometric function. A numerical evaluation of these expressions agrees to a very high level of precision with the result obtained above from the numerical integration.

Moreover, these particular hypergeometric functions can be expressed in terms of complete elliptic integrals [25]

$$(1 - z^2)^3 F[\frac{1}{2}, \frac{5}{2}; 4; 1 - z^2] = \frac{32}{15\pi} \left[z^2(z^2 - 9)K(\sqrt{1 - z^2}) + (3 + 7z^2 - 2z^4)E(\sqrt{1 - z^2}) \right] \quad ,$$

$$(1 - z^2)^5 F[\frac{1}{2}, \frac{7}{2}; 6; 1 - z^2] = \frac{512}{945\pi} (1 - z^2)^2 \left[(45z^8 - 7z^6 + 22z^4 - 11z^2 + 15)K(\sqrt{1 - z^2}) \right. \\ \left. - (15z^8 + 31z^6 + 12z^4 - 17z^2 + 23)E(\sqrt{1 - z^2}) \right] \quad , \quad (4.6)$$

where $K(k)$ and $E(k)$ are complete elliptic integrals of the first and second kind, respectively.

The power spectrum in the asymptotic scaling regime is dominated by the long-wavelength modes and is given by

$$S(k, t) = \frac{1}{N} \langle \vec{\Phi}_{\vec{k}}(t) \cdot \vec{\Phi}_{-\vec{k}}(t) \rangle = \mathcal{Z} \frac{\eta^3}{g} \left[\frac{J_{p+1}(k\eta)}{(k\eta)^{p+1}} \right]^2 \quad (4.7)$$

where $p = 1, 2$ for radiation dominated or matter dominated, respectively and where \mathcal{Z} is a constant of order one. At long times $\eta \gg 1$ this power spectrum becomes strongly peaked at $k = 0$ and receives contribution only from a narrow region $k \leq 1/\eta$ as evidenced by figs. (5-7). Since the sum rule constrains the total integral of the power spectrum to be

$$\int k^2 dk S(k, t) = \text{constant} \quad (4.8)$$

we are led to the conclusion that at asymptotically long times the power spectrum is sharply peaked at $k = 0$ becoming asymptotically a delta function. This is the signal of the formation of a zero momentum condensate, much in the same manner as observed in the case of Minkowski space-time [6,11] and also in dynamical critical phenomena in condensed matter systems [8,9].

B. Equation of state

The expectation value of the energy momentum tensor in the non-equilibrium density matrix is of the fluid form in terms of the energy density ε and pressure P given by

$$\frac{\lambda}{N} \varepsilon = \frac{1}{2} \left[m_0^4 - \left(\frac{\lambda}{2N} \langle \vec{\Phi} \cdot \vec{\Phi} \rangle \right)^2 \right] + \frac{\lambda}{2} \int d^3k \coth \left[\frac{\omega_k}{2T_i} \right] \left[|\dot{\varphi}_k(t)|^2 + W_k^2(t) |\varphi_k(t)|^2 \right] \quad (4.9)$$

$$\frac{\lambda}{N} (P + \varepsilon) = \lambda \int d^3k \coth \left[\frac{\omega_k}{2T_i} \right] \left[|\dot{\varphi}_k(t)|^2 + \frac{k^2}{3a^2(t)} |\varphi_k(t)|^2 \right] \quad (4.10)$$

where $\varphi_k(t)$ obeys the equations of motion (2.24) with $W_k^2(t)$ given by eq. (2.13) and we had set to zero the coupling to the Ricci scalar. Using the equations of motion (2.24) and the definition of the self-consistent frequencies $W_k^2(t)$ (2.13), it can be easily verified that ε and P satisfy the covariant conservation law

$$\dot{\varepsilon}(t) + 3 \frac{\dot{a}}{a} [\varepsilon(t) + P(t)] = 0 \quad (4.11)$$

Passing on to conformally rescaled mode functions and conformal time and writing the asymptotic forms of the scale factor in radiation dominated and matter dominated dominated cosmologies as

$$C(\eta) = \left(\frac{\eta}{2} \right)^p \quad \text{with } p = 1 \text{ for Radiation dominated ; } p = 2 \text{ for Matter dominated} \quad (4.12)$$

we will focus our discussion on the behavior of the pressure and the energy density in the scaling regime wherein the conformally rescaled mode functions in conformal time are given by eq. (3.14, 3.15), which can be handily written as

$$f_k(t) = A_0 \eta^{p+3/2} \frac{J_{p+1}(x)}{x^{p+1}} \quad (4.13)$$

in either case. The sum rule (3.10) with vanishing coupling to the Ricci scalar leads to the following identity in the asymptotic regime

$$C^2(\eta) \left[-m_0^2 + \frac{\lambda}{2N} \langle \vec{\Phi} \cdot \vec{\Phi} \rangle - \frac{C''(\eta)}{C^3(\eta)} \right] \rightarrow -\frac{(2p+3)(2p+1)}{4\eta^2} + \mathcal{O}(1/\eta^4) \quad (4.14)$$

up to perturbatively small corrections. The leading term in this sum rule is given by the sum rule (3.5) which upon using the asymptotic scaling mode functions (3.14)-(3.15) can be written in the following compact form

$$|A_0|^2 g \int dx \left[\frac{J_{p+1}(x)}{x^p} \right]^2 = 2^{-2p} \quad (4.15)$$

Using this result and after some straightforward algebra we find that the asymptotic behavior of the energy density and pressure are given by

$$\frac{\lambda}{N} \varepsilon = \frac{|A_0|^2 g 2^{2p}}{C^2(\eta) \eta^2} \int x^2 dx \left\{ \left[\left(\frac{1}{2} - p \right) \frac{J_{p+1}(x)}{x^{p+1}} + \frac{J'_{p+1}(x)}{x^p} \right]^2 + x^2 \left[\frac{J_{p+1}(x)}{x^{p+1}} \right]^2 \right\} \quad (4.16)$$

$$\frac{\lambda}{N} (\varepsilon + P) = \frac{|A_0|^2 g 2^{2p+1}}{C^2(\eta) \eta^2} \int x^2 dx \left\{ \left[\left(\frac{1}{2} - p \right) \frac{J_{p+1}(x)}{x^{p+1}} + \frac{J'_{p+1}(x)}{x^p} \right]^2 + \frac{x^2}{3} \left[\frac{J_{p+1}(x)}{x^{p+1}} \right]^2 \right\} \quad (4.17)$$

where the prime stands for derivative with respect to the argument. The integral is independent of conformal time and the only dependence on η is in the prefactors, the product

$$C^2(\eta) \eta^2 \propto \begin{cases} C^4(\eta) & \text{for Radiation dominated} \\ C^3(\eta) & \text{for Matter dominated} \end{cases} \quad (4.18)$$

thus we find that in either case

$$\varepsilon \propto \rho_{back}(\eta) \quad (4.19)$$

where $\rho_{back}(\eta)$ is the *background* energy density in either radiation dominated ($\rho_{back}(\eta) \propto C^{-4}(\eta)$) or matter dominated ($\rho_{back}(\eta) \propto C^{-3}(\eta)$). Since ε and P satisfy the covariant conservation equation and so do the background energy density and pressure we conclude that the fluid resulting from the fluctuations of the scalar field in the asymptotic scaling regime *obeys the same equation of state as the background fluid*. Indeed careful evaluation of the integrals using the properties of the Gamma function [24] as analytic functions of the variable p lead to the following remarkable results

$$\frac{\lambda}{N} \varepsilon(\eta) C^2(\eta) \eta^2 = 3 p \frac{p+1/4}{p-1} \quad (4.20)$$

$$\frac{\lambda}{N} [\varepsilon(\eta) + P(\eta)] C^2(\eta) \eta^2 = 2 (p + 1) \frac{p + 1/4}{p - 1} \quad (4.21)$$

leading to one of the important results of this article

$$\frac{P}{\varepsilon} = \frac{1}{3} \left(\frac{2}{p} - 1 \right) = \begin{cases} \frac{1}{3} & \text{for Radiation dominated } (p = 1) \\ 0 & \text{for Matter dominated } (p = 2) \end{cases} \quad (4.22)$$

The simple poles at $p = 1$ in eqs. (4.20)-(4.21) reflect ultraviolet logarithmic singularities in the integrals which have been evaluated with an upper limit taken to infinity. However we remark that the scaling form requires that the upper limit be of order $x \ll 1/\eta$ so that the coefficient of the Bessel functions can be replaced by momentum independent constants leading to true scaling solutions. A numerical evaluation of the integrand in (4.16)-(4.17) reveals that these are strongly peaked near $x = 0$ and most of the contribution arises from the interval $x \leq 5$ even for $p = 1$. The results (4.20)-(4.21) are the analytic continuation of these integrals in the variable p taking the upper cutoff to infinity. The variable p thus acts as a regulator much in the same way as analytic or dimensional regularization and the equation of state is insensitive to the regularization procedure.

Defining $\varepsilon = \delta\rho$ with ε the energy density given by eq.(4.16) and $\delta\rho$ as the contribution to the energy density from the fluctuations of the scalar field, the fact that $\delta\rho/\rho_{back} = \text{constant}$ i.e, independent of time is the statement that the power spectrum for the density fluctuations is of the Harrison-Zeldovich form, i.e., scale invariant, for long-wavelength scalar density perturbations [1,2].

C. Universality of scaling

Although we have studied radiation and matter dominated cosmologies separately while the most physical scenario involves a smooth transition between the two regimes we now argue that if the transition between the two regimes occurs over a very long time scale our analysis leads to the conclusion that scaling will be a robust feature in the regimes in which the scale factor is dominated by a power law. In the asymptotic regime, when the effective time dependent mass vanishes, the sum rule

$$g \int_0^1 q^2 dq |f_q(\eta)|^2 = C^2(\eta) \quad (4.23)$$

is fulfilled for an arbitrary scale factor, not necessarily a power law, however if $C(\eta)$ is *not* a power law, the solution is *not* of the scaling form.

Consider the scale factor in the general case in which the background fluid has a radiation and a matter component as given by (2.52), i.e,

$$C(\eta) = 1 + H_i \eta + \frac{H_i^2 r_i}{4(1 + r_i)} \eta^2 \quad (4.24)$$

in a realistic scenario the ratio of the initial matter density to the initial radiation density is $r_i \ll 1$. Therefore in the regime $\frac{1}{r_i} \gg H_i \eta \gg 1$ the scale factor is approximately that

of a radiation dominated cosmology and therefore a power law, hence the sum rule above (4.23) leads to the scaling form of the solution. During the time scales of the crossover between radiation and matter domination $r_i H_i \eta \approx 1$ the scale factor is *not* a pure power law. The sum rule above *still holds* but it does not entail a scaling form for the mode functions. However for $H_i \eta r_i \gg 1$ again the scale factor is a power law now describing a matter dominated cosmology and the sum-rule again leads to a scaling form for the solution in the long-wavelength limit. However when the new scaling form emerges the solution *does not* depend on the past history, i.e. on the violations of scaling during the crossover or the previous scaling solution in the radiation dominated regime. The reason for this is that the sum rule above is *local* in time and fixes the absolute value of the coefficient A_0 and the scaling form in terms of the Bessel function, however the *phase* of the coefficients A_q and certainly of A_0 will depend on the history and will *not* be determined by the sum-rule. However in the semiclassical stochastic description advocated in the previous section this phase is treated as a stochastic variable with a Gaussian distribution function. Hence all of the initial information and the history contained in the phases of the scaling solution can be treated stochastically for the long-wavelength components.

V. COMPARISON WITH THE $O(N)$ NON LINEAR SIGMA MODEL:

There are many similarities but also many differences with the results obtained in the case of the *classical* non-linear sigma model in refs. [15,16]. The non-linear sigma model describes the non-linear interactions of Goldstone bosons in a broken symmetry phase. To begin with let us establish an important difference: the non-linear sigma model *cannot* be used to describe the *phase transition* because the fields are constrained to the vacuum manifold. Hence all of the details of the dynamics of the phase transition, the linear instabilities, growth of long-wavelength modes, the different time scales and regimes and the explanation of why the power spectrum is peaked at long-wavelength as a consequence of the early exponential growth of long-wavelength fluctuations simply cannot be captured by the non-linear sigma model. These details that are deeply dependent on the details of the phase transition have important consequences: the amplitude of the mode functions in the scaling regime is *non-perturbatively large* in the weak coupling limit since $A_0 \propto 1/\sqrt{g}$. This amplitude is not arbitrary, if \hbar is restored, the initial amplitudes of the mode functions are of $\mathcal{O}(\sqrt{\hbar})$, the non-perturbative growth of the amplitude of long-wavelength fluctuations is a consequence of the linear exponential instabilities which are a dynamical hallmark of the phase transition. Furthermore the corrections to scaling embodied in the contribution from the Neumann functions in eqs.(3.12)-(3.13) have very precise coefficients B_q since these are constrained by the Wronskian condition *and the self-consistency condition* (3.18). Both the coefficients A_q ; B_q carry information from the initial conditions because of the constancy of the Wronskian, the long wavelength limit A_0 ; B_0 are of order $\mathcal{O}(1/\sqrt{g})$ as a consequence of the self-consistency conditions (3.16)-(3.18) and the phases can only be determined from the numerical evolution.

There are also very important similarities: in the asymptotic regime the scaling form of the solution (3.14)-(3.15) is the *same* in the model studied here and in the $O(N)$ non-linear sigma model [15,16]. In the non-linear sigma model the coefficient of the Bessel function is fixed by the length of the $O(N)$ vector, this length is interpreted as the vacuum expectation

value of the field. In fact in the model studied here the sum-rule (3.5) can be identified with the same constraint. The similarity of the asymptotic scaling solutions is a consequence of the fact that eq.(3.5) is local as discussed above.

The description of the non-equilibrium dynamics via the evolution of the quantum density matrix leads to a clear interpretation of the emergence of a semiclassical stochastic description of long-wavelength fluctuations at long times. Our analysis thus provides a consistent *microscopic, quantum field theoretical derivation* for the classical stochastic treatment of the Goldstone modes in the asymptotic region which was used in references [15,16]. Furthermore, our detailed analysis quantifies the time scales for which such semiclassical stochastic description is valid and goes beyond providing corrections to scaling.

In ref. [15] the energy density of the scalar field in the asymptotic scaling regime was found to be proportional to that of the background in the matter dominated case and differing from that of the background by a weak logarithmic dependence on η in the radiation dominated regime. The logarithmic dependence has a counterpart in our result in the form of the pole in the energy density and pressure as $p \rightarrow 1$ in the expressions (4.20)-(4.21) obtained via an analytic continuation in the variable p , as we pointed out the equation of state is independent of the regularization. Furthermore we emphasized that both the energy density and the pressure are dominated by modes $k \ll 1/\eta$ i.e. *superhorizon* modes.

VI. CONCLUSIONS AND FURTHER QUESTIONS

In this article we have studied in detail the non-equilibrium dynamics of a symmetry breaking phase transition in a spatially flat radiation and or matter dominated FRW background. Anticipating the necessity for a non-perturbative treatment, we studied the linear sigma model with a scalar field in the vector representation of $O(N)$ in leading order in the large N limit. The Liouville equation for the quantum density matrix is solved in this limit from which we extract the necessary correlation functions to leading order. The advantage of working with a density matrix is that this description allows a clear interpretation of the emergence of a semiclassical description in terms of a field probability distribution functional.

The main goal of our study is to provide a thorough understanding the process of phase ordering beginning from a state of local thermodynamic equilibrium at an initial temperature larger than the critical, the cosmological expansion triggers the phase transition when the effective time dependent temperature falls below the critical. The key issues that we address in this article are the emergence of a scaling regime and of a characteristic dynamical correlation length that determines the spatial extent of the correlated regions of broken symmetry and the consequences of such scaling behavior upon the equation of state of the field fluctuations.

The non-equilibrium dynamics after the phase transition is characterized by three distinct time scales: during the early stage after the phase transition the dynamics is dominated by the exponential growth of long-wavelength fluctuations associated with spinodal instabilities. This is essentially a linear regime in which the backreaction can be ignored for weak self-coupling and an analytic description is available. An intermediate time scale is defined when the backreaction from the self-consistent mean-field is comparable to the tree level terms in the equations of motion and determines the onset of non-perturbative and

non-linear dynamics which must be studied numerically. A third, asymptotic time scale reveals the emergence of a scaling regime for radiation or matter dominated FRW backgrounds. This stage is dominated by large amplitude fluctuations with wavelengths of order of or larger than the causal horizon. This regime is characterized by a dynamical physical correlation length $\xi_{phys} = d_H(t)$ with $d_H(t)$ the size of the causal horizon and the onset of a non-equilibrium condensate at zero momentum. The dynamical correlation length determines the size of the correlated domains inside which the field fluctuations probe the broken symmetry states, hence there is exactly one correlated domain per causal horizon. The field correlations vanish for distances larger than $d_H(t)$ by causality. In this regime the phases of the long-wavelength quantum mode functions become time independent and their amplitude becomes non-perturbatively large, the approach via the density matrix reveals the emergence of a semiclassical but stochastic description and provides a microscopic justification for a semiclassical stochastic treatment in the asymptotic regime.

A remarkable corollary of the scaling solution of the equations of motion is that in the asymptotic regime the equation of state of the fluid described by the fluctuations of the scalar field is the same as that of the background fluid that drives the dynamics of the scale factor. An important consequence of this behavior of the fluid is that $\delta\rho/\rho_{back} = \text{constant}$ which results in a Harrison-Zeldovich spectrum of scalar density perturbations for superhorizon wavelengths [1,2].

The self-consistency of the equations of motion entails very precise corrections to scaling. We argue that the scaling solution is a universal feature of a scale factor that is a power law and is independent of the crossover between the radiation and matter dominated regimes.

Some important aspects remain to be explored further. A thorough investigation of cosmological perturbations has been performed in ref. [16]. An important aspect of the models under consideration is that the perturbations in the energy momentum tensor are *non-linear* in terms of the field fluctuations, unlike the case of scalar field perturbations in the inflationary stage. The correction to the energy density is given by the expectation value of the energy momentum tensor of the fluctuations and even to leading order in the large N limit, this is a quadratic form in terms of the fluctuations [see eq.(4.9)]. This feature has an important consequence in the spectrum of primordial density perturbations: decoherence effects tend to suppress the acoustic peaks at large angular momentum l (small angular resolution) (see [16] for a clear treatment in the non-linear sigma model). As we have argued above, there are very precise and important corrections to scaling that are manifest for subhorizon wavelengths, i.e. $k \gg 1/\eta$. An important possibility is that these corrections to scaling affect the correlations of the energy momentum tensor and therefore the power spectrum of scalar density perturbations on small angular scales. This would arise from the interference effects between the scaling contributions and those of the scaling corrections, much in the same manner as the next-to-leading contribution in the sum rules discussed above. As we have highlighted these interference effects are necessary for the self-consistency of the method leading to a precise form of the scaling corrections. These corrections had not been taken into account in the analysis of the power spectrum at small angular scales and whether they lead to significant corrections and or a dramatic change of the picture of decoherence effects in causal perturbations is an open question that merits careful study.

Another important aspect that requires further understanding is the description of the

non-equilibrium dynamics in the case of a single scalar field which cannot be addressed reliably in the large N expansion. In the large N limit an important feature of the phase diagram of the theory is that the coexistence and the spinodal line merge as a consequence of the Ward identity associated with the continuous symmetry [13,14] and the long-wavelength physics described by the dynamics of Goldstone bosons. Static renormalization group arguments in scalar field theories lead to a Maxwell constructed free energy with a region of phase coexistence that joins the minima of the free energy and no metastable region [22,23] for all values of N with no basic difference between the cases of a single scalar field or a multiplet of fields. It is important to find a consistent non-perturbative approximation scheme for $N = 1$ to study the *dynamics* in this case. Another important aspect to be studied further is whether the scaling solution survives $1/N$ corrections, in particular it is clear that in next to leading order in the large N approximation another time scale associated with collisional processes should emerge and the relevant question is whether this microscopic scale will modify substantially the scaling solution and in particular the equation of state. These and other related questions are currently under consideration.

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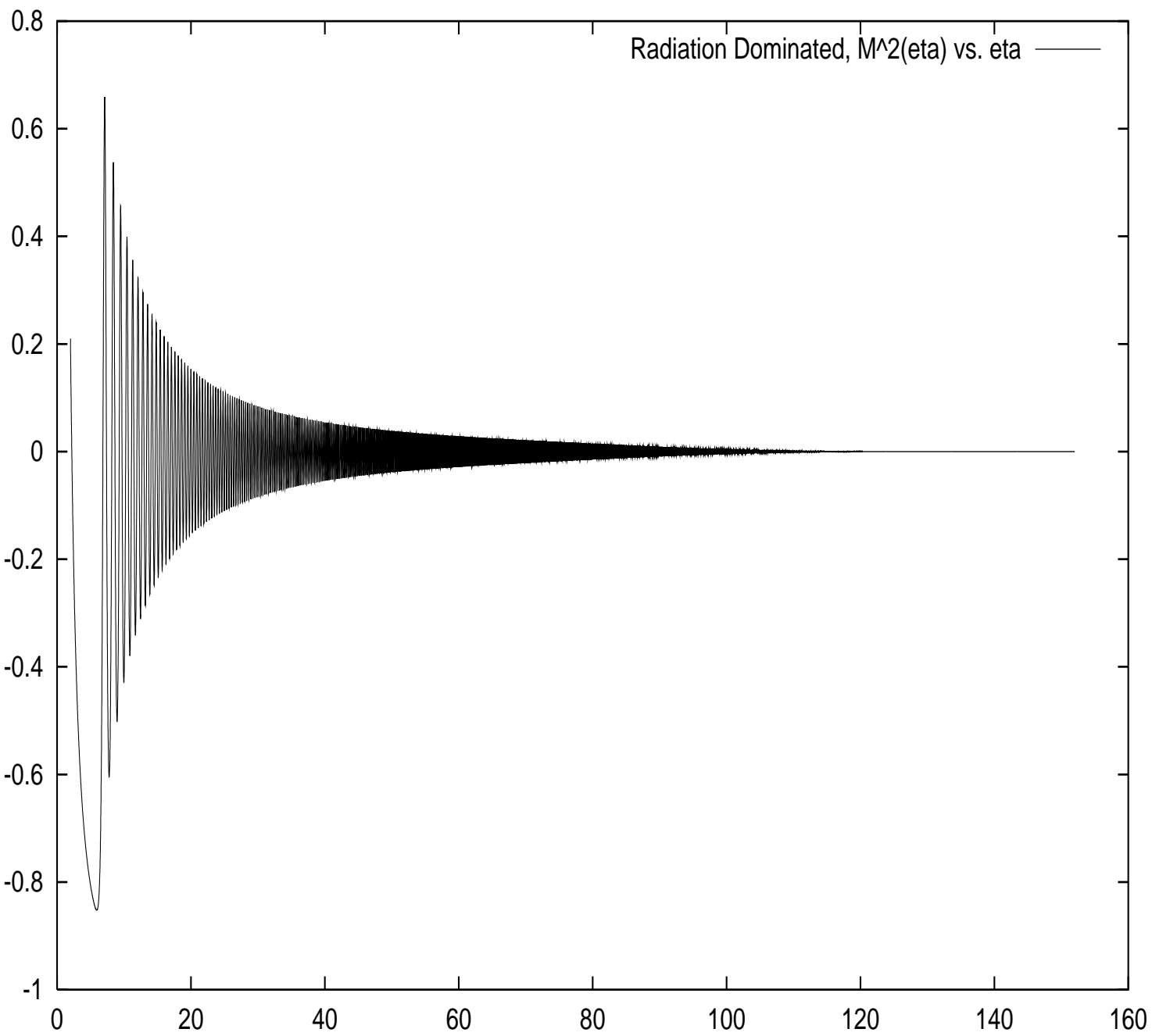


FIG. 1. $M^2(\eta)$ vs. η (conformal time in units of m_R^{-1}) for $T_c^k = 1.1$, $g = 10^{-5}$. Radiation dominated universe.

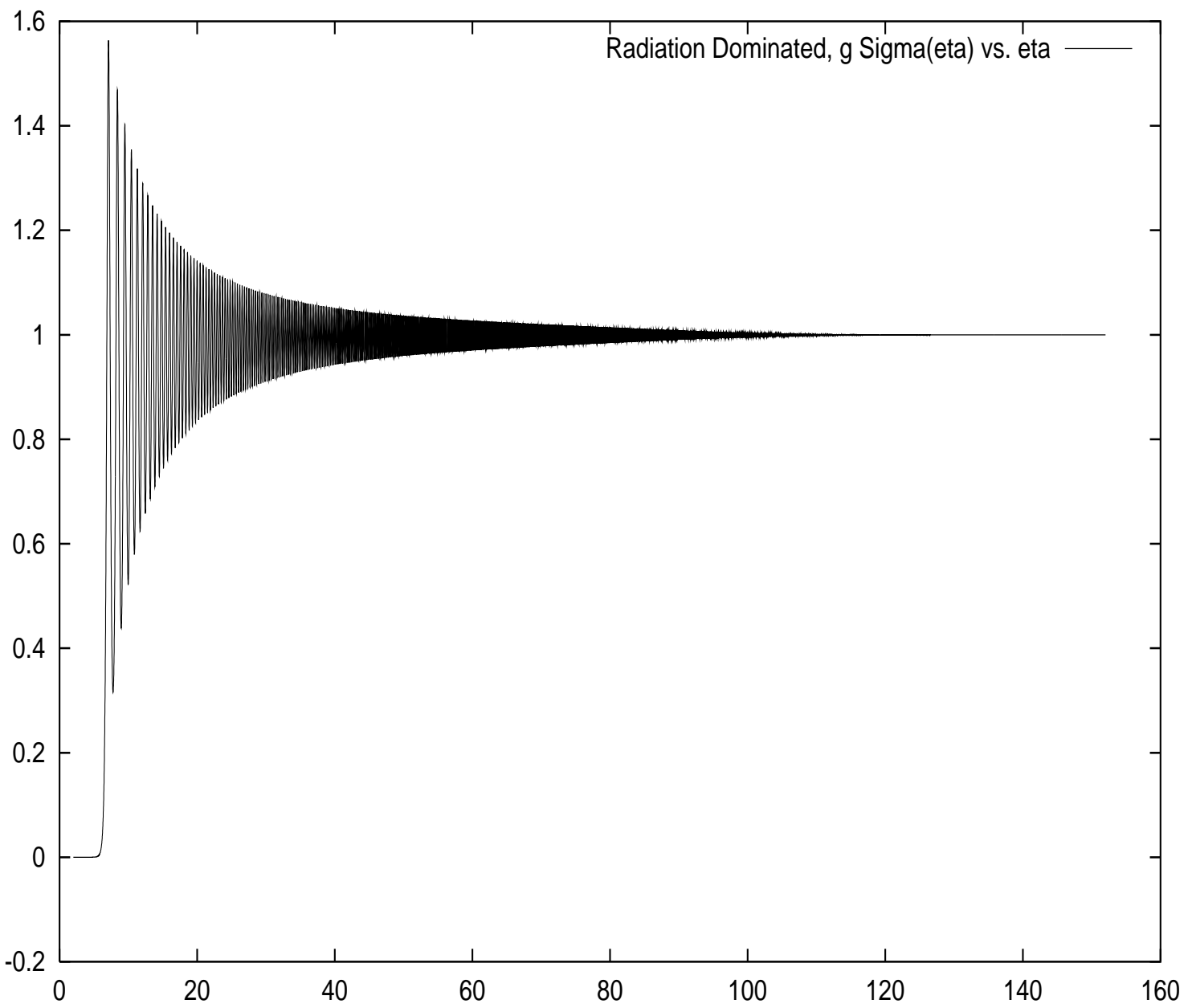
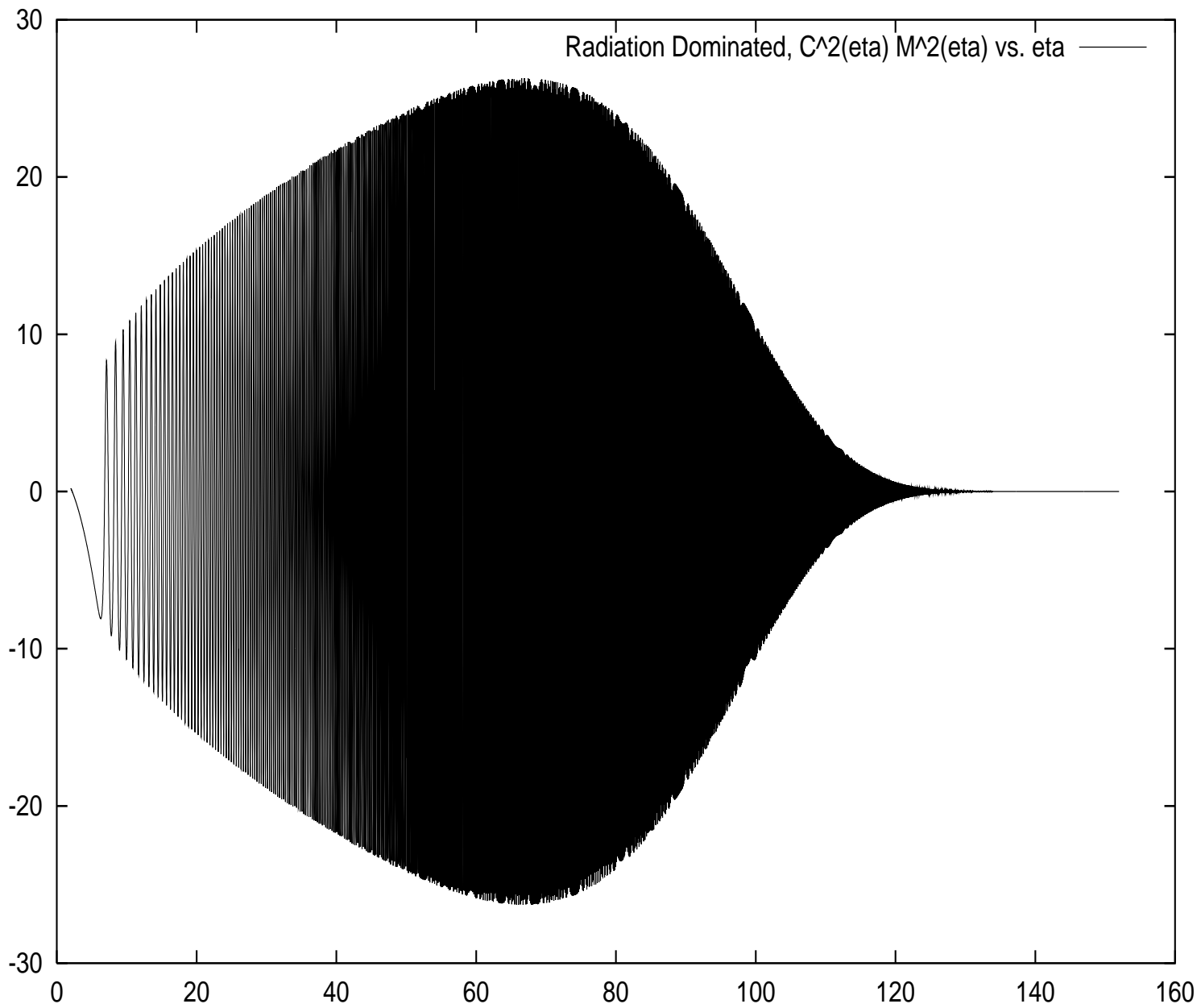


FIG. 2. $g\Sigma(\eta)$ vs. η (conformal time in units of m_R^{-1}) for $T_c/T_e = 3$, $g = 10^{-5}$. Radiation dominated universe.

FIG. 3. $C^2(\eta)M^2(\eta)$ vs. η (conformal time in units of m_R^{-1}) for $T_c = 1.1$, $g = 10^{-5}$. Radiation dominated universe.



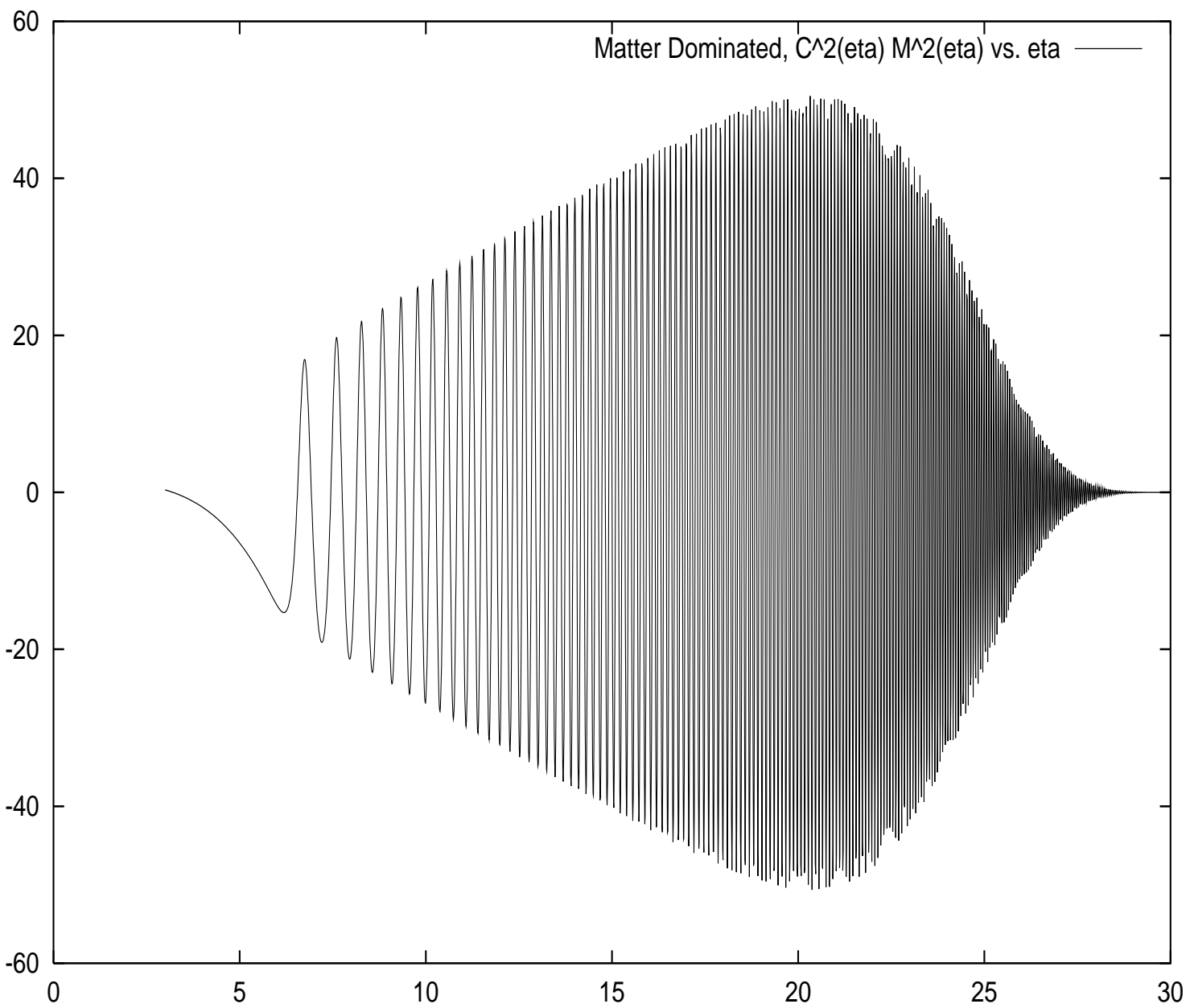


FIG. 4. $C^2(\eta)M^2(\eta)$ vs. η (conformal time in units of m_R^{-1}) for $\frac{T_i}{T_c} = 1.1$, $g = 10^{-4}$. Matter dominated universe.

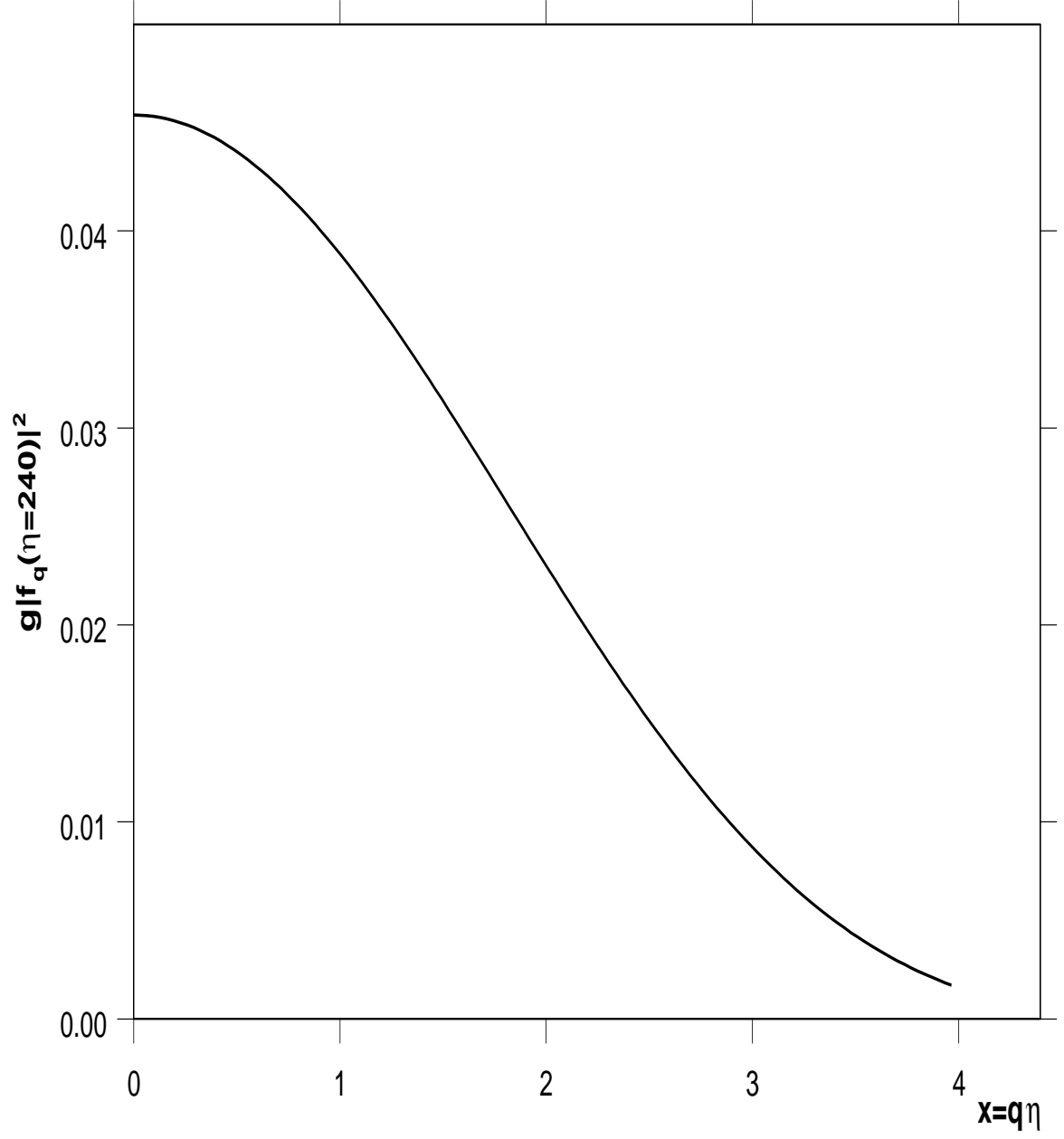


FIG. 5. $g\eta^{-5}|f_q(\eta=240)|^2$ vs. $x=q\eta$ for $\frac{T_i}{T_c}=3$, $g=10^{-5}$. Radiation dominated universe.

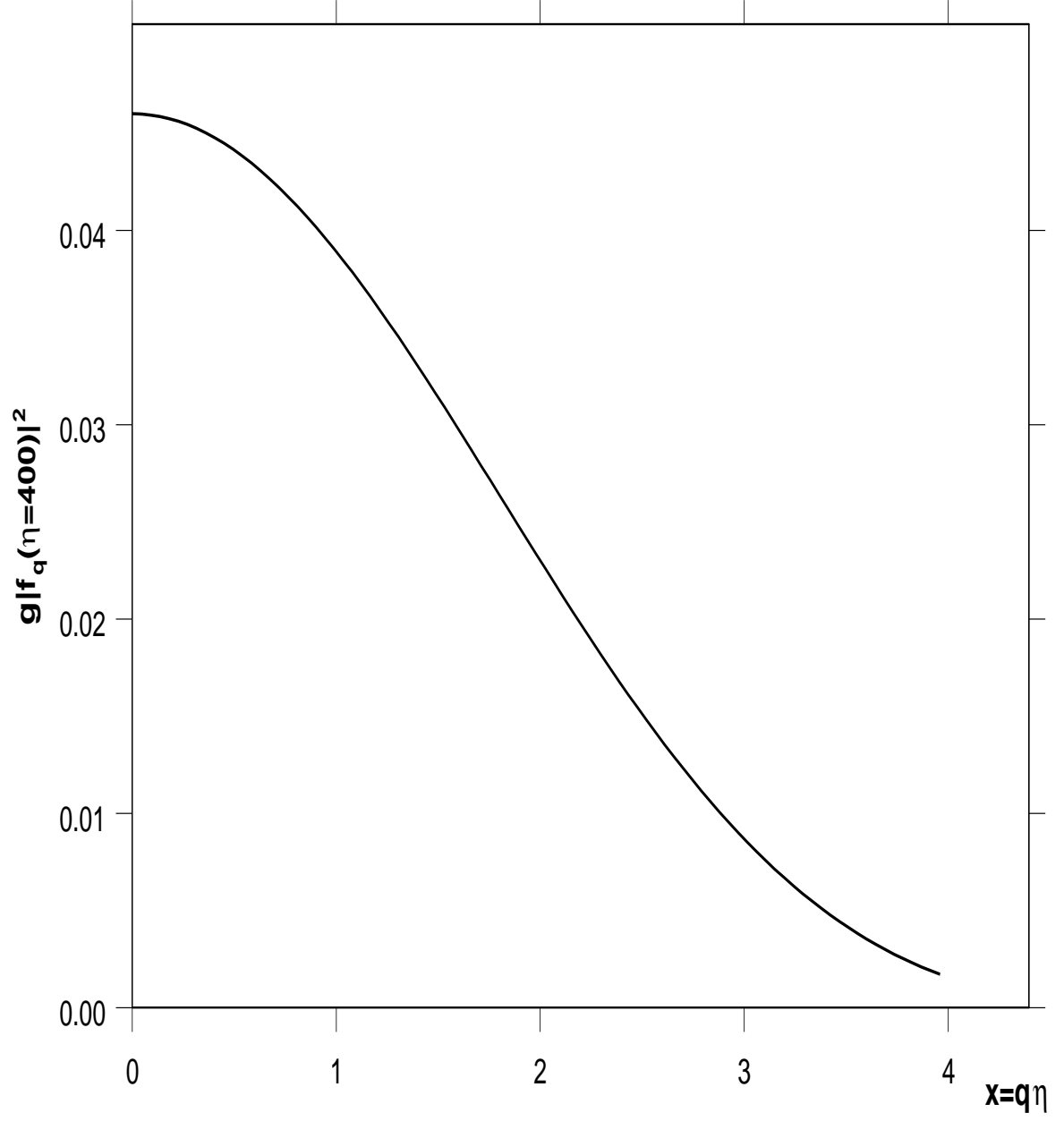


FIG. 6. $g\eta^{-5}|f_q(\eta=400)|^2$ vs. $x=q\eta$ for $\frac{T_i}{T_c}=3$, $g=10^{-5}$. Radiation dominated universe.

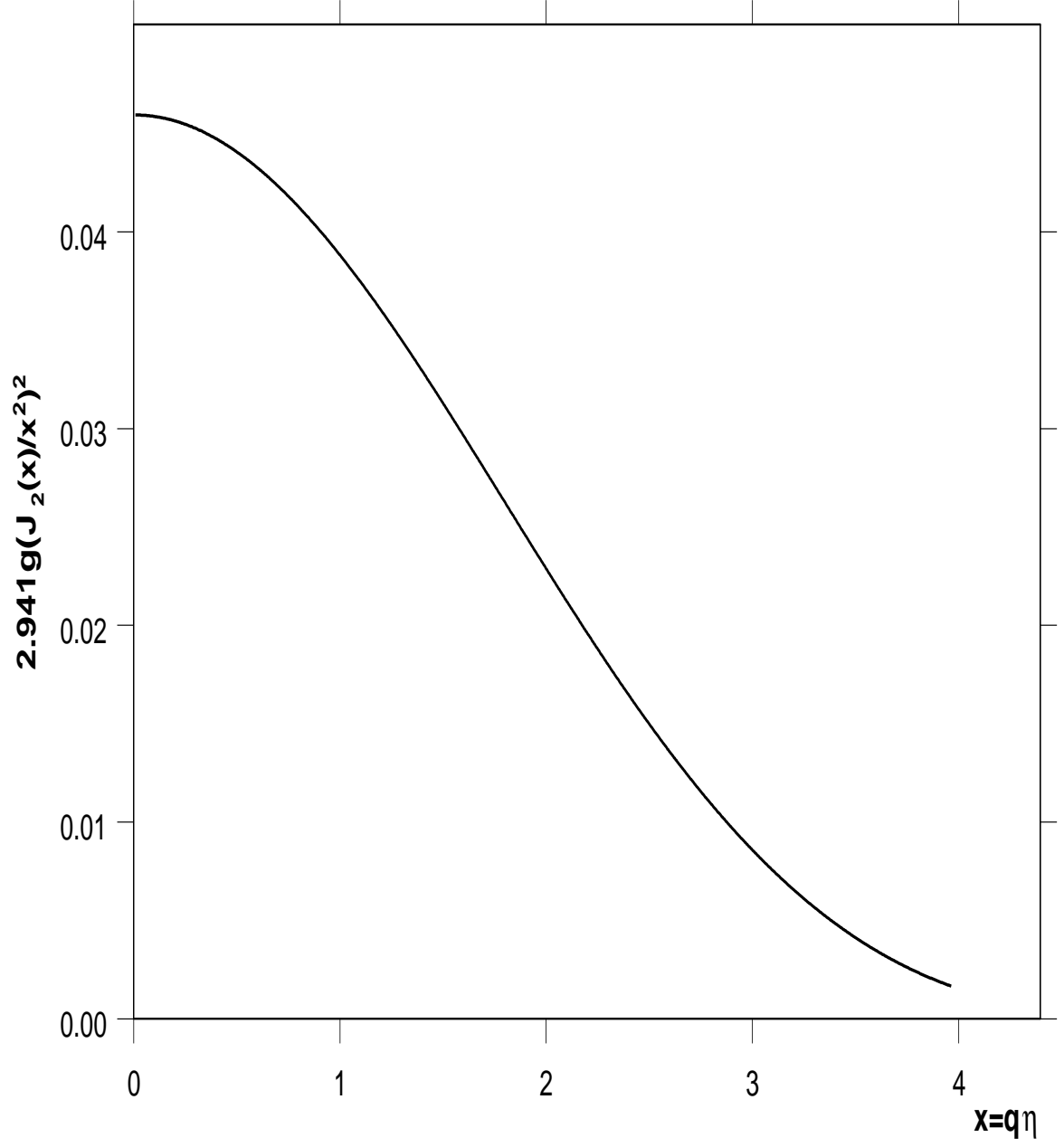


FIG. 7. $g|A_0 J_2(x)/x^2|^2$ vs. $x = q\eta$ for $\frac{T_i}{T_c} = 3$, $g = 10^{-5}$. Radiation dominated universe.

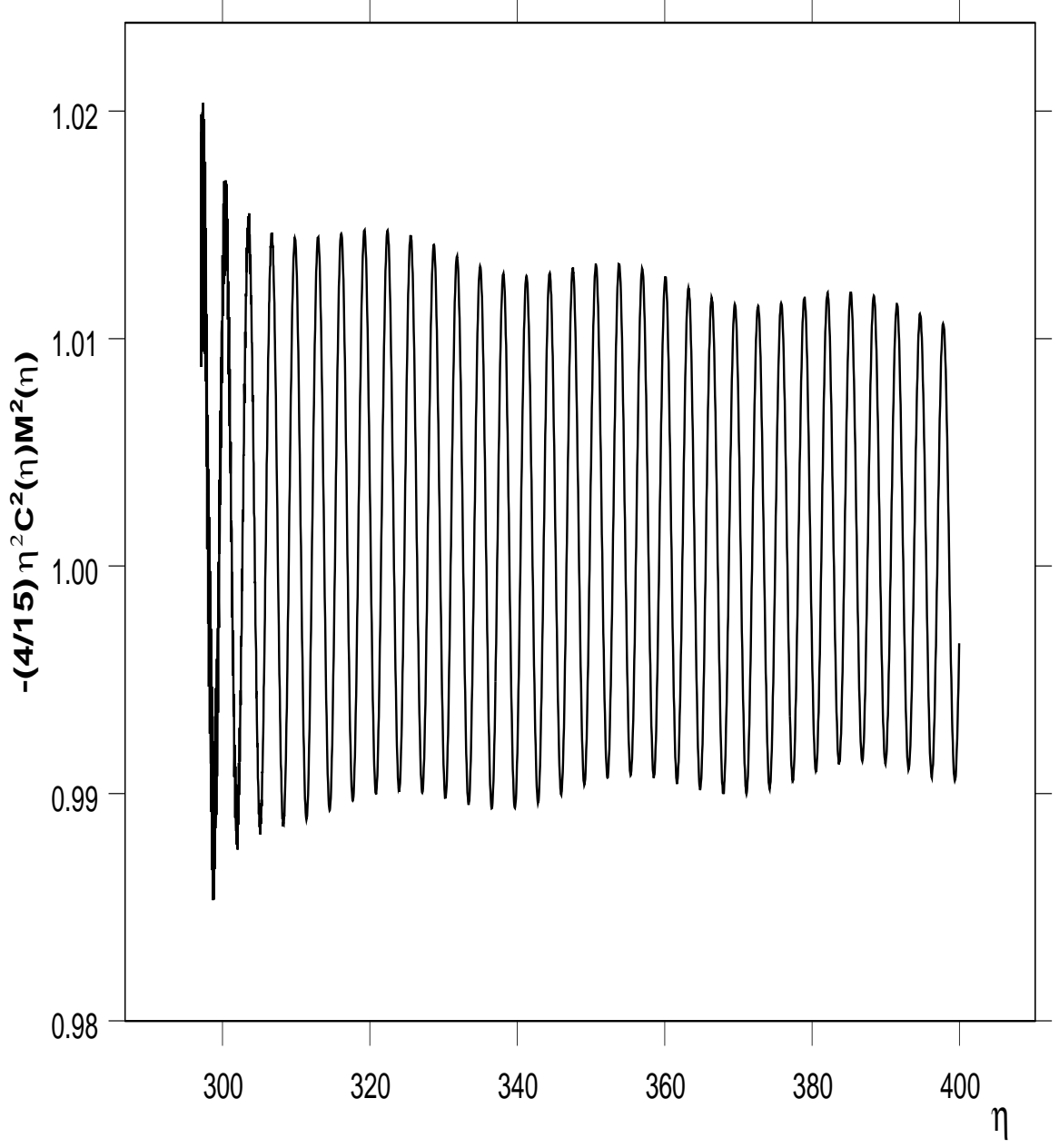


FIG. 8. $(-4/15)\eta^2 C^2(\eta)M^2(\eta)$ vs. η (in units of m_R^{-1} for $\frac{T_i}{T_c} = 3$, $g = 10^{-5}$. Radiation dominated universe.

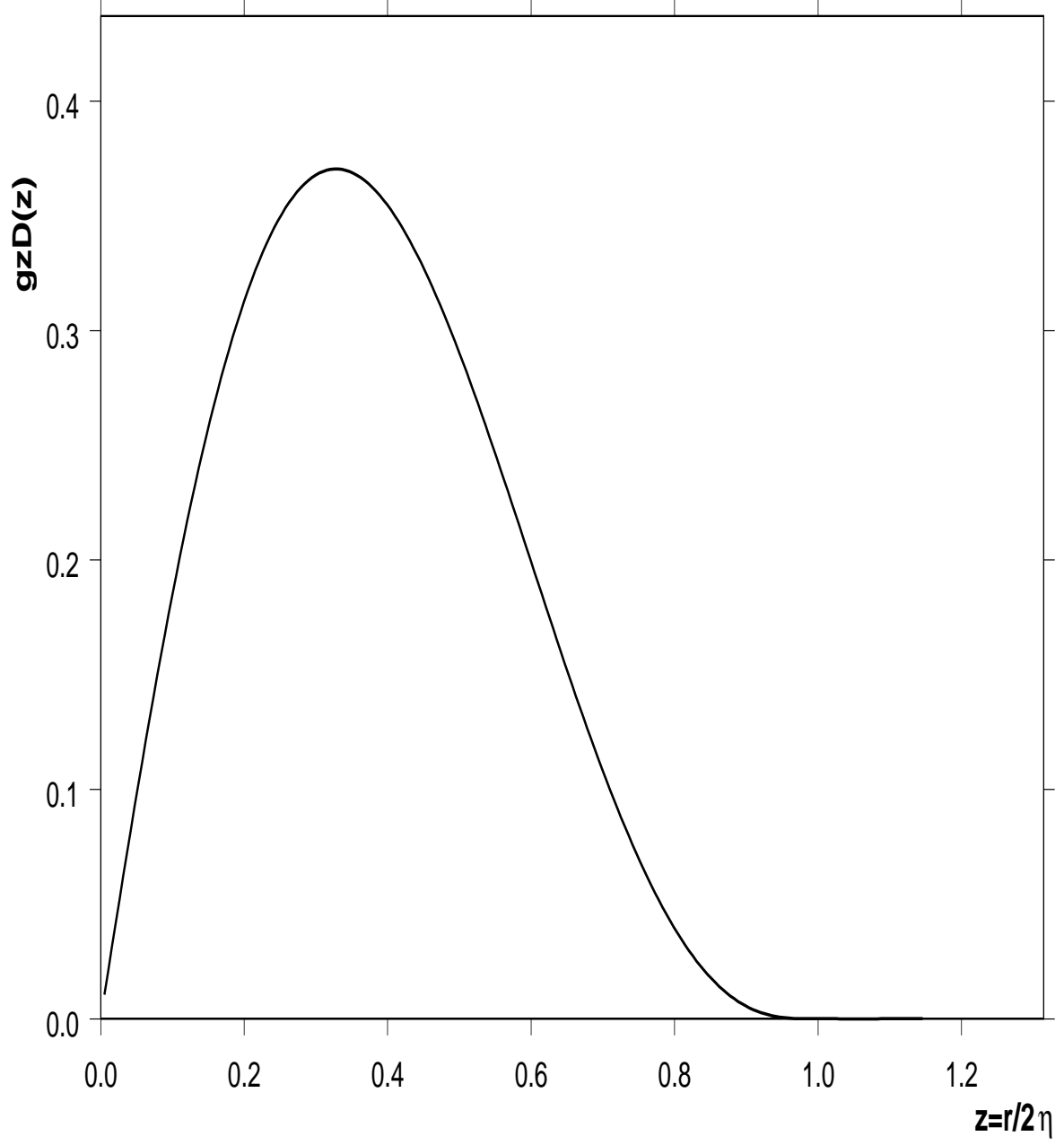


FIG. 9. $gzD(z)$ vs. $z = r/2\eta$ at $\eta = 240$ (in units of m_R) for $\frac{T_i}{T_c} = 3$, $g = 10^{-5}$. Radiation dominated universe.

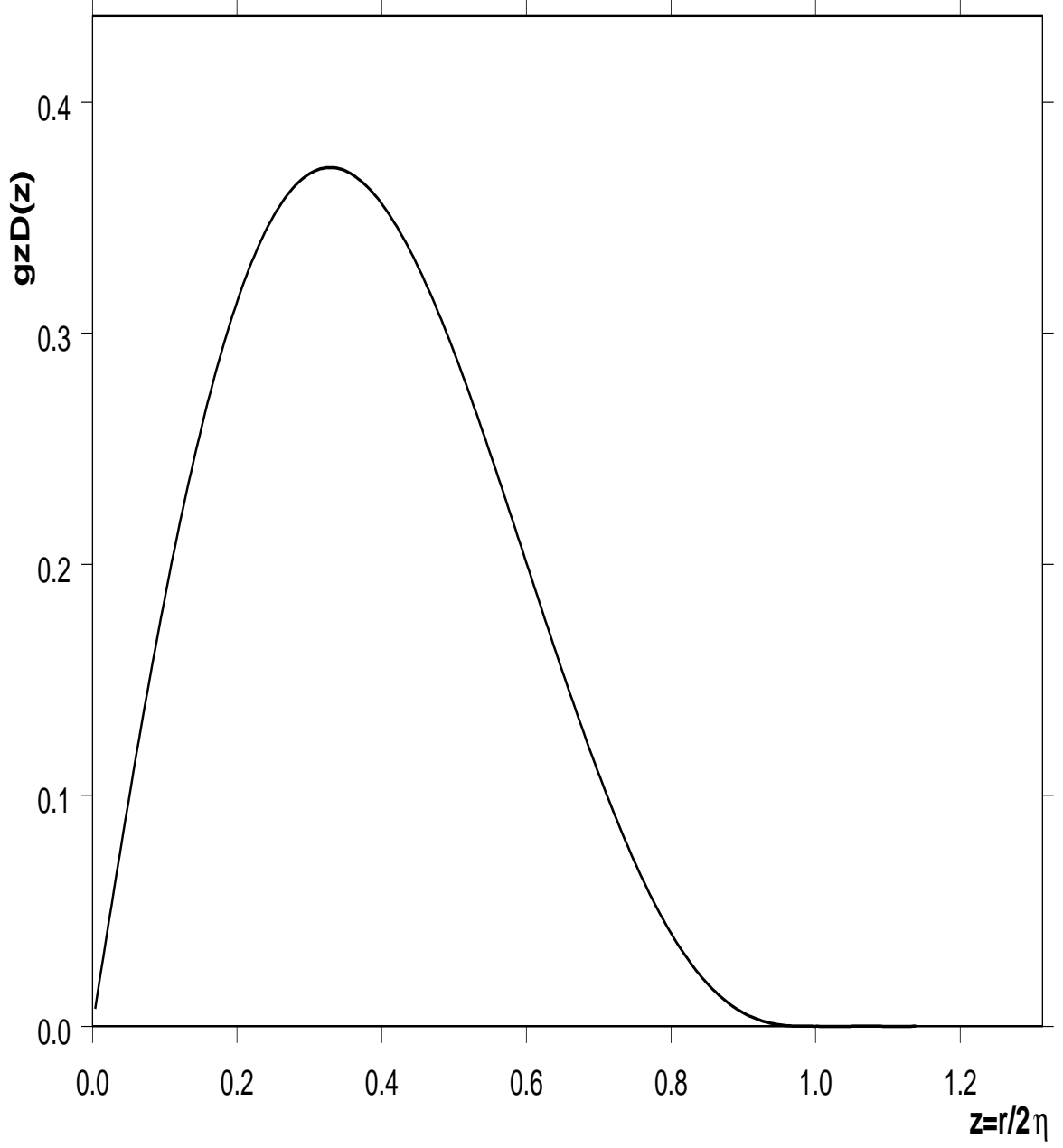


FIG. 10. $gzD(z)$ vs. $z = r/2\eta$ at $\eta = 400$ (in units of m_R) for $\frac{T_i}{T_c} = 3$, $g = 10^{-5}$. Radiation dominated universe.

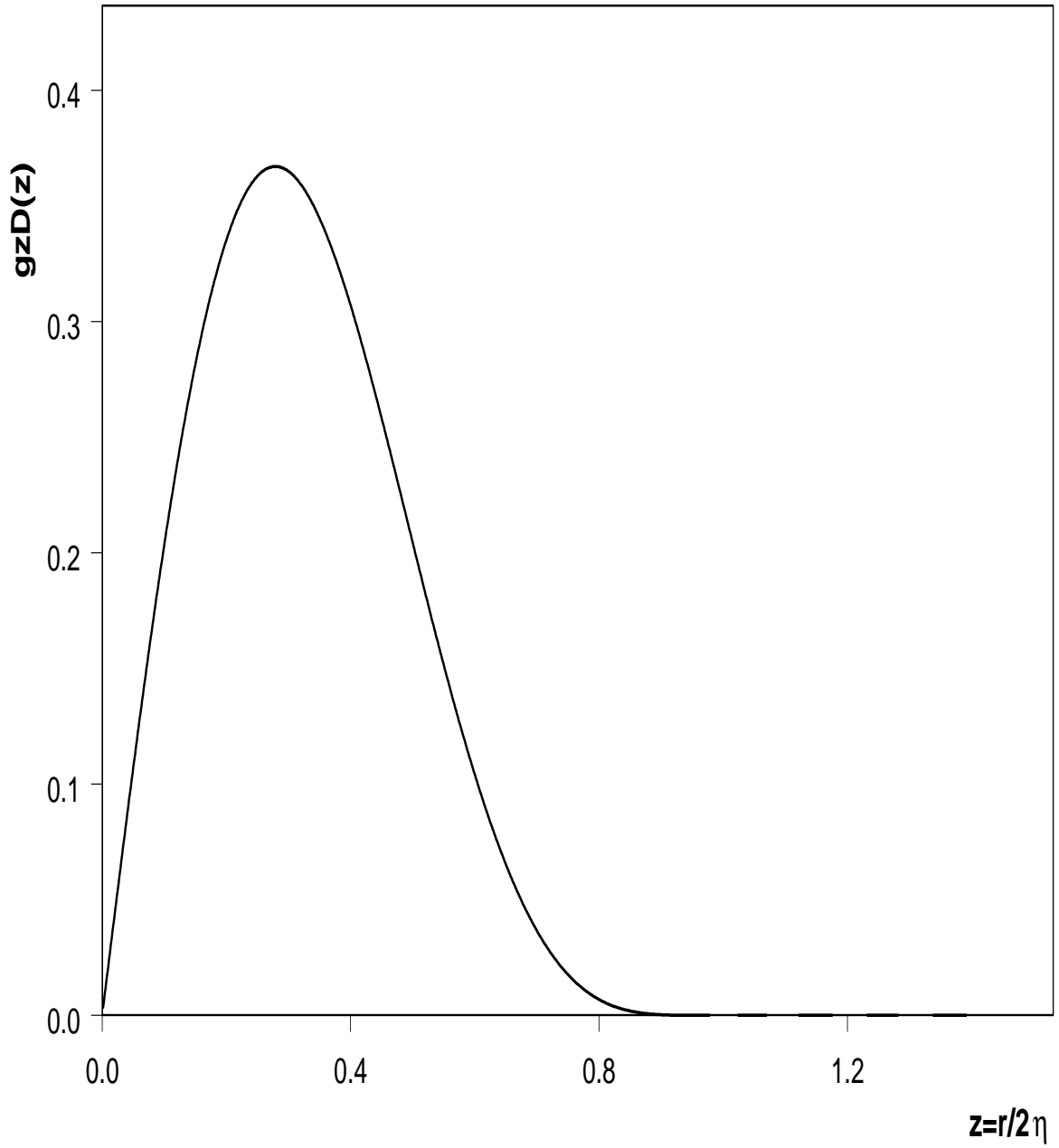


FIG. 11. $gzD(z)$ vs. $z = r/2\eta$ at $\eta = 400$ (in units of m_R) for $\frac{T_i}{T_c} = 3$, $g = 10^{-5}$. Matter dominated universe.